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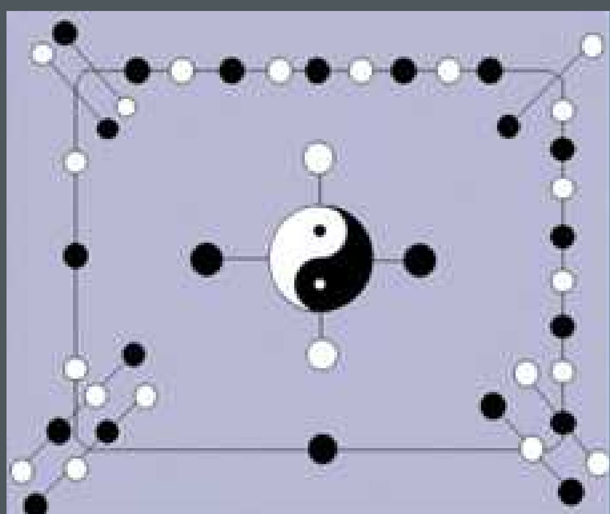

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By Abraham Lincoln, an American president.

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Smarandache-Zagreb Index on Three Graph Operators

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Abstract: Many researchers have studied several operators on a connected graph in which one make an attempt on subdivision of its edges. In this paper, we show how the Zagreb indices, a particular case of Smarandache-Zagreb index of a graph changes with these operators and extended these results to obtain a relation connecting the Zagreb index on operators.

Key Words: Subdivision graph, ladder graph, Smarandache-Zagreb index, Zagreb index, graph operators.

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§1. Introduction

A single number that can be used to characterize some property of the graph of a molecule is called a topological index. For quite some time interest has been rising in the field of computational chemistry in topological indices that capture the structural essence of compounds. The interest in topological indices is mainly related to their use in nonempirical quantitative structure property relationships and quantitative structure activity relationships. The most elementary constituents of a (molecular) graph are vertices, edges, vertex-degrees, walks and paths [14]. They are the basis of many graph-theoretical invariants referred to (somewhat imprecisely) as topological index, which have found considerable use in Zagreb index.

Suppose $G = (V, E)$ is a connected graph with the vertex set V and the edge set E. Given an edge $e = \{u, v\}$ of G. Now we can define the *subdivision graph* $S(G)$ [2] as the graph obtained from G by replacing each of its edge by a path of length 2, or equivalently by inserting an additional vertex into each edge of G.

In [2], Cvetkocic defined the operators $R(G)$ and $Q(G)$ are as follows:

the operator $R(G)$ is the graph obtained from G by adding a new vertex corresponding to each edge of G and by joining each new vertex to the end vertices of the edge corresponding to it. The operator $Q(G)$ is the graph obtained from G by inserting a new vertex into each edge of G and by joining edges those pairs of these new vertices which lie on the adjacent edges of G (See also [16]).

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The Wiener index W belongs among the oldest graph-based structure-descriptors topological indices [12,17]. Numerous of its chemical applications were reported in [6,11] and its mathematical properties are well known [3]. Another structure-descriptor introduced long time ago [4] is the Zagreb index M_1 or more precisely, the first Zagreb index, because there exists also a second Zagreb index M_2. The research background of the Zagreb index together with its generalization appears in chemistry or mathematical chemistry.

In this paper, we concentrate on Zagreb index [8] with a pair of topological indices denoted $M_1(G)$ and $M_2(G)$ [1,9,10,13,18]. The *first Zagreb index*

$$M_1(G) = \sum_{u \in V(G)} d^2(u),$$

and the *second Zagreb index*

$$M_2(G) = \sum_{uv \in E(G)} d(u)d(v).$$

Generally, let G be a graph and H its a subgraph. The *Smarandache-Zagreb index of G relative to H* is defined by

$$M^S(G) = \sum_{u \in V(H)} d^2(u) + \sum_{(u,v) \in E(G \setminus H)} d(u)d(v).$$

Particularly, if $H = G$ or $H = \emptyset$, we get the first or second Zagreb index $M_1(G)$ and $M_2(G)$, respectively.

A *Tadpole graph* [15] $T_{n,k}$ is a graph obtained by joining a cycle graph C_n to a path of length k and a *wheel graph* W_{n+1} [7] is defined as the graph $K_1 + C_n$, where K_1 is the singleton graph and C_n is the cycle graph [8]. A ladder graph $L_n = K_2 \Box P_n$, where P_n is a path graph. For all terminologies and notations not defined in here, we refer to Harary [5].

§2. A relation connecting the Zagreb indices on $S(G)$, $R(G)$ and $Q(G)$ for the Tadpole graph and Wheel graph

We derive a relation connecting the Zagreb index with the subdivision graph $S(G)$ and two graph operators $R(G)$ and $Q(G)$, where, n, k are integers≥ 1 in this section.

Theorem 2.1 *The first Zagreb index*

$$M_1(S(T_{n,k})) = M_1(T_{n,k}) + 4(n+k).$$

Proof The Tadpole graph $T_{n,k}$ contains $n+k-2$ vertices of degree 2, one vertex of degree 3 and a pendent vertex. Hence $M_1(T_{n,k}) = 4n+4k+2$. The subdivision graph $S(T_{n,k})$ contains $n+k$ additional subdivision vertices. Hence

$$M_1(S(T_{n,k})) = M_1(T_{n,k}) + 4(n+k)$$

$$M_1(S(T_{n,k})) = 8n + 8k + 2. \tag{2.1}$$

□

Theorem 2.2 $M_1(R(T_{n,k})) = M_1(S(T_{n,k})) + 6(2n + 2k + 1)$.

Proof Each vertex v of degree l in $T_{n,k}$ is of degree $2l$ in $R(T_{n,k})$ and all the subdivision vertices in $S(T_{n,k})$ is of the same degree l in $R(T_{n,k})$. So,

$$M_1(R(T_{n,k})) = 16(n-1) + 16(k-1) + 4(n+k) + 40$$

$$M_1(R(T_{n,k})) = M_1(S(T_{n,k})) + 6(2n+2k+1) \tag{2.2}$$

from equation (2.1). □

Theorem 2.3 $M_1(Q(T_{n,k})) = \begin{cases} M_1(T_{n,k}) + 2M_1(S(T_{n,k})) + 14, & \textit{if } k = 1; \\ M_1(T_{n,k}) + M_1(S(T_{n,k})) + 16, & \textit{if } k \geq 2. \end{cases}$

Proof If $k = 1$, the graph $Q(T_{n,k})$ contains the sub graph $T_{n,k}$. The $n + k - 2$ subdivision vertices of degree 2 in $S(T_{n,k})$ are of double the degree in $Q(T_{n,k})$ and only 2 vertices of degree 5. So,

$$\begin{aligned} M_1(Q(T_{n,k})) &= 16(n + k - 2) + 50 + M_1(T_{n,k}) \\ &= 2(8n + 8k + 2) + M_1(T_{n,k}) + 14. \end{aligned}$$

Hence $M_1(Q(T_{n,k})) = M_1(T_{n,k}) + 2M_1(S(T_{n,k})) + 14$ if $k = 1$.

For $k \geq 2$, the $n+k-4$ subdivision vertices of degree 2 in $S(T_{n,k})$ is of degree 4 in $Q(T_{n,k})$ and only 3 vertices of degree 5 and one vertex of degree 3. Hence

$$M_1(Q(T_{n,k})) = M_1(T_{n,k}) + 16(n + k) + 20$$

and

$$M_1(Q(T_{n,k})) = M_1(T_{n,k}) + M_1(S(T_{n,k})) + 16, \text{if} \quad k \geq 2.$$

□

Theorem 2.4 $M_2(S(T_{n,k})) = \begin{cases} 2M_2(T_{n,k}) - 2, & \textit{if } k = 1; \\ 2M_2(T_{n,k}) - 4, & \textit{if } k \geq 2 \end{cases}$

Proof Among the $n + k$ vertices in $T_{n,k}$, only one vertex of degree 1, one vertex of degree 3 and $n + k - 2$ vertices of degree 2, among which the $n + k - 4$ pairs of vertices of degree 2, the three pairs of vertices of degree 2 and 3 and a pair of vertices of degree 2 and 1 are adjacent with each other for $k \geq 2$. Hence, $k \geq 2$,

$$M_2(T_{n,k}) = 4n + 4k + 4. \tag{2.3}$$

For $k = 1$, the $n - 1$ vertices of degree 2, one vertex of degree 3 and a pendent vertex among which there will be $n - 2$ pairs of vertices of degree 2, two pairs of vertices of degree 2 and 3 and a pair of vertices of degree 3 and 1 are adjacent with each other. So when $k = 1$,

$$M_2(T_{n,k}) = 4n + 7. \tag{2.4}$$

The new $n+k$ vertices of degree 2 is inserted in $T_{n,k}$ to construct $S(T_{n,k})$.

$$M_2(S(T_{n,k})) = 4(2n-2) + 4(2k-2) + 20 = 8n + 8k + 4 \tag{2.5}$$

Hence $M_2(S(T_{n,k})) = 2M_2(T_{n,k}) - 4$, for $k \geq 2$, from equation (2.3).

$$M_2(S(T_{n,k})) = 2M_2(T_{n,k}) - 2,$$

for $k = 1$, from equation (2.4). □

Theorem 2.5 $M_2(R(T_{n,k})) = \begin{cases} 4M_2(S(T_{n,k})) + 4, & \text{if } k = 1; \\ 4M_2(S(T_{n,k})) + 8, & \text{if } k \geq 2. \end{cases}$

Proof If $k = 1$, the $n-2$ pairs of vertices of degree 4, $2n-2$ pairs of vertices of degree 2 and 4, two pairs of vertices of degree 4 and 6, four pairs of vertices of degree 2 and 6 and a pair of vertices of degree 2 are adjacent to each other. So, $M_2(R(T_{n,k})) = 16(n-2)+8(2k-2)+8(2n-2)+100$. Hence

$$M_2(R(T_{n,k})) = 32n+32k+36 = 4M_2(S(T_{n,k}))+4. \tag{2.6}$$

if $k = 1$, from equation (2.5).

The vertices which are of degree 1 in $T_{n,k}$ are of degree $2l$ in $R(T_{n,k})$ and all the subdivision vertices in $S(T_{n,k})$ remains unaltered in $R(T_{n,k})$. In $R(T_{n,k})$, the $n+k-4$ pairs of vertices of degree 4, $2n-1$ pairs of degree 4 and 2, three pairs of vertices of degree 4 and 6, three pairs of vertices of degree 2 and 6 and one pair of vertices of degree 2 are adjacent to each other in $R(T_{n,k})$ when $k \geq 2$. Hence

$$M_2(R(T_{n,k})) = 16(n-2) + 8(2n-2) + 16(k-2) + 8(2k-2) + 120,$$

$$M_2(R(T_{n,k})) = 32n + 32k + 24, \tag{2.7}$$

$$M_2(R(T_{n,k})) = 4(8n + 8k + 4) + 8 = 4M_2(S(T_{n,k})) + 8,$$

if $k \geq 2$ from equation (2.5). □

Theorem 2.6 $M_2(Q(T_{n,k})) = \begin{cases} M_2(R(T_{n,k})) + 39, & \text{if } k = 1; \\ M_2(R(T_{n,k})) + 46, & \text{if } k = 2; \\ M_2(R(T_{n,k})) + 47, & \text{if } k \geq 3. \end{cases}$

Proof We divide the proof of this theorem into three cases.

Case 1: When $k = 1$, the $n-3$ pairs of vertices of degree 4, $2n-4$ pairs of vertices of degree 2 and 4, one pair of vertices of degree 5, two pairs of vertices of degree 2 and 5, two pairs of vertices of degree 3 and 5, a pair of vertices of degree 3 and 4, a pair of vertices of 4 and 1, and four pairs of vertices of degree 4 and 5 are adjacent to each other in $Q(T_{n,k})$. Hence

$$\begin{aligned} M_2(Q(T_{n,k})) &= 16(n-3) + 8(2n-4) + 91 = 32n + 91 \\ &= (32n + 16k + 36) + 39) = M_2(R(T_{n,k})) + 39 \end{aligned}$$

from equation (2.6).

Case 2: When $k = 2$, the $n-3$ pairs of vertices of degree 4, $2n-4$ pairs of vertices of degree 2 and 4, three pair of vertices of degree 5, three pairs of vertices of degree 2 and 5, 4 pairs of vertices of degree 3 and 5, a pair of vertices of degree 1 and 3, a pair of vertices of 2 and 3, and two pairs of vertices of degree 4 and 5 are adjacent to each other in $Q(T_{n,k})$. Hence

$$\begin{aligned} M_2(Q(T_{n,k})) &= 16(n-3) + 8(2n-4) + 214 = 32n + 134 \\ &= (32n + 32k + 24) + 46 = M_2(R(T_{n,k})) + 46 \end{aligned}$$

from equation (2.7).

Case 3: When $k \geq 3$, there are $n + k - 6$ pairs of vertices of degree 4, $2n + 2k - 8$ pairs of vertices of degree 2 and 4, three pairs of vertices of degree 5, three pairs of vertices of degree 2 and 5, three pairs of vertices of degree 3 and 5, a pair of vertices of degree 3 and 1, a pair of vertices of degree 2 and 3, a pair of vertices of degree 4 and 3 and three pairs of vertices of degree 4 and 5 are neighbours to each other in $Q(T_{n,k})$, with which,

$$\begin{aligned} M_2(Q(T_{n,k})) &= 16(n + k - 6) + 8(2n + 2k - 8) + 231 = 32n + 32k + 71 \\ &= (32n + 32k + 24) + 47 = M_2(R(T_{n,k})) + 47 \end{aligned}$$

from equation (2.7). □

Theorem 2.7 *For the wheel graph W_{n+1}, $M_1(S(W_{n+1})) = M_1(W_{n+1}) + 8n$.*

Proof In W_{n+1}, it has n vertices of degree 3 and one vertex, the center of wheel of degree n. So,

$$M_1(W_{n+1}) = 9n + n^2. \tag{2.8}$$

By inserting a vertex in each edge of W_{n+1}, $M_1(S(W_{n+1})) = M_1(W_{n+1}) + 8n$.

$$M_1(S(W_{n+1})) = n^2 + 17n. \tag{2.9}$$

□

Theorem 2.8 $M_1(R(W_{n+1})) = 4M_1(S((W_{n+1}))) - 24n$.

Proof The degrees of the subdivision vertices in $S(W_{n+1})$ remains unaltered in $R(W_{n+1})$ and a vertex of degree l in W_{n+1}, is of degree $2l$ in $R(W_{n+1})$.

$$\begin{aligned} M_1(R(W_{n+1})) &= 4n^2 + 44n = 4(n^2 + 17n) - 24n \\ &= 4M_1(S((W_{n+1}))) - 24n. \end{aligned} \tag{2.10}$$

□

Theorem 2.9 $M_1(Q(W_{n+1})) = M_1(R((W_{n+1}))) + M_1(W_{n+1}) + n(n+1)^2$.

Proof Clearly $Q(W_{n+1})$ contains the subgraph W_{n+1}. Every subdivision vertex on the edges of the subgraph C_n in $S(W_{n+1})$ is adjacent with the four subdivision vertices, two on the spoke and two on the edges of C_n. Each of the subdivision vertex on the edges of C_n is of degree 6. Also every subdivision vertex on a spoke is adjacent with the $n-1$ subdivision vertices on the other spokes and is adjacent with 2 subdivision vertices on the edges of C_n with which the subdivision vertex on the spoke is of degree $n+3$. Therefore,

$$\begin{aligned} M_1(Q(W_{n+1})) &= M_1(W_{n+1}) + 36n + (n+3)^2 n \\ &= M_1(W_{n+1}) + (4n^2 + 44n) + (n^3 + 2n^2 + n) \end{aligned}$$

and

$$M_1(Q(W_{n+1})) = M_1(R((W_{n+1}))) + M_1(W_{n+1}) + n(n+1)^2$$

by equation (2.10). □

Theorem 2.10 $M_2(S(W_{n+1})) = M_2(W_{n+1}) + (9n - n^2)$.

Proof A vertex of degree 3 is adjacent with two vertices of degree 3 and with the hub of the wheel so that

$$M_2(W_{n+1}) = 3n^2 + 9n \tag{2.11}$$

In $S(W_{n+1})$, the $2n$ additional subdivision vertices are inserted. A vertex of degree 3 is adjacent with three vertices of degree 2 and all the subdivision vertices on the spoke are adjacent to the hub.

$$\begin{aligned} M_2(S(W_{n+1})) &= 2n^2 + 18n = (3n^2 + 9n) + (9n - n^2) \\ &= M_2(W_{n+1}) + (9n - n^2) \end{aligned} \tag{2.12}$$

from equation (2.11). □

Theorem 2.11 $M_2(R(W_{n+1})) = 4M_2(S((W_{n+1}))) + 8n^2$.

Proof The degrees of the subdivision vertices in $S(W_{n+1})$ remains the same in $R(W_{n+1})$ and every vertex in W_{n+1} is of double the degree in $R(W_{n+1})$. Every vertex of degree 6 is adjacent with the hub, two vertices of degree 6 and three subdivision vertices. The subdivision vertices on the spoke is adjacent with the hub. Hence

$$\begin{aligned} M_2(R(W_{n+1})) &= 72n + 16n^2 = 4(2n^2 + 18n) + 8n^2 \\ &= 4M_2(S((W_{n+1}))) + 8n^2 \end{aligned} \tag{2.13}$$

from equation (2.12). □

Theorem 2.12 *For a wheel graph W_{n+1},*

$$M_2(Q(W_{n+1})) = \frac{2M_2(R((W_{n+1})) + 3M_2(S(W_{n+1})) + (n^4 + 7n^3 + n^2 + 27n)}{2}.$$

Proof Every subdivision vertex in $S(W_{n+1})$(other than the subdivision vertices on the spoke) is of degree 6 and is adjacent with the the two vertices of degree 3, two vertices of degree 6, two vertices of degree $n+3$. A vertex of degree 3 is adjacent with the subdivision vertices on the spokes of degree $n+3$, and the subdivision vertices on the spoke is adjacent with the hub of the wheel and the $n-1$ subdivision vertices on the remaining spokes.

$$\begin{aligned} M_2(Q(W_{n+1})) &= \left[36+36+12(n+3)+3(n+3)+n(n+3)+\frac{((n+3)^2(n-1)}{2}\right]\times n \\ &= \frac{2(16n^2+72n)+3(2n^2+18n)+(n^4+7n^3+n^2+27n)}{2} \\ &= \frac{2M_2(R((W_{n+1}))+3M_2(S(W_{n+1}))+(n^4+7n^3+n^2+27n)}{2} \end{aligned}$$

by applying equations (2.12) and (2.13). □

§3. A relation connecting the Zagreb indices on $S(G)$, $R(G)$ and $Q(G)$ for the Ladder graph

In this section, we assume n being an integer≥ 3. When $n=1$, L_1 is the path P_1 and When $n=2$, L_2 is the cycle C_4 for which the the relations on the Zagreb index are same as in the case of P_k and C_n respectively.

Theorem 3.1 *For the ladder graph L_n, $M_1(S(L_n)) = M_1(L_n) + 4(3n-2)$.*

Proof The ladder graph L_n contains $2n-4$ vertices of degree 3 and four vertices of degree 2. So

$$M_1(L_n) = 18n - 20 \tag{3.1}$$

Since there are $3n-2$ edges in L_n there is an increase of $3n-2$ subdivision vertices in $S(L_n)$.

$$M_1(S(L_n)) = M_1(L_n)+4(3n-2) = 30n-28. \tag{3.2}$$

□

Theorem 3.2 $M_1(R(L_n)) = 2M_1(S(L_n)) + (24n-32)$.

Proof The subdivision vertices in $S(L_n)$ retains the same degree in $R(L_n)$ and a vertex of degree l in L_n is of degree $2l$ in $R(L_n)$. Hence,

$$M_1(R(L_n)) = 2^2(3n-2) + 72(n-2) + 64$$

and

$$M_1(R(L_n)) = 84n-88 = 2M_1(S(L_n))+(24n-32) \tag{3.3}$$

from equation (3.2). □

Theorem 3.3 $M_1(Q(L_n)) = M_1(R(L_n)) + 42n - 88$.

Proof The graph $Q(L_n)$ contains the subgraph L_n. The subdivision vertices on the top and the bottom of the ladder say v_1 and v_k in $Q(L_n)$ is of degree 4 corresponding to the adjacencies and the nearest subdivision vertices of v_1 and v_k are of degree 5 corresponding to the 3 adjacent subdivision vertices in $S(L_n)$. The remaining $3n-8$ subdivision vertices are of degree 6. So

$$M_1(Q(L_n)) = M_1(L_n) + 132 + 6^2(3n-8) = M_1(R(L_n)) + 42n - 88,$$

from equation (3.3). □

Theorem 3.4 $M_2(S(L_n)) = M_2(L_n) + 9n$.

Proof In L_n, two vertices of degree 2 are adjacent with a vertex of degree 3 and a vertex of degree 2. The $2n-8$ pairs of vertices of degree 3 are adjacent with the vertex of degree 2. Hence,

$$M_2(L_n) = 32 + 18(n-3) + 9(n-2) = 27n - 40. \quad (3.4)$$

In $S(L_n)$, eight pairs of vertices of degree 2, $6n-12$ pairs of vertices of degree 2 and three are adjacent to each other. So

$$M_2(S(L_n)) = 32 + 6(6n-12) = M_2(L_n) + 9n \quad (3.5)$$

from equation (3.4). □

Theorem 3.5 $M_2(R(L_n)) = 5M_2(S(L_n)) - 40$.

Proof The degrees of the subdivision vertices in $S(L_n)$ is unaffected in $R(L_n)$, and all the vertices in L_n become double the degree in $R(L_n)$. In $R(L_n)$, eight pairs of vertices of degree 4 and 2, $6n-12$ pairs of vertices of degree 2 and 6, two pairs of vertices of degree 4, $3n-8$ pairs of vertices of degree 6 , four pairs of vertices of degree 4 and six are adjacent to each other. So,

$$M_2(R(L_n)) = 180n - 240 = 5M_2(S(L_n)) - 40 \quad (3.6)$$

from equation (3.5). □

Theorem 18 $M_2(Q(L_n))= \begin{cases} 2M_2(R(L_n)) + (-36n-44), & \textit{if } n=3; \\ M_2(R(L_n)) + (-72n+548), & \textit{if } n=4; \\ 2M_2(R(L_n)) + (-36n-4), & \textit{if } n \geq 5. \end{cases}$

Proof We divide the proof of this result into three cases following.

Case 1: If $n=3$, the $Q(L_n)$ contains the subgraph L_n. In $Q(L_n)$, there are four pairs of vertices of degree 4 and 5, four pairs of vertices of degree 4 and 2, four pairs of vertices of degree 2 and 5, four pairs of vertices of degree 5 and 6 , two pairs of vertices of degree 5, four pairs of vertices of degree 5 and 3, $6n-16$ pairs of vertices of degree 3 and 6, the $6n-18$ pairs of vertices of vertices of degree 6 are adjacent to each other. Hence,

$$M_2(Q(L_3)) = 382 + 18(6n-16) + 36(6n-18)$$

and

$$M_2(Q(L_n)) = 2M_2(R(L_n)) + (-36n - 44)$$

from equation (3.6).

Case 2: If $n = 4$, four pairs of vertices of degree 4 and 5, four pairs of vertices of degree 4 and 2, four pairs of vertices of degree 2 and 5, eight pairs of vertices of degree 5 and 6, four pairs of vertices of degree 5 and 3, four pairs of vertices of degree 3 and 6 and $6n - 20$ pairs of vertices of degree 6 are adjacent to each other in $Q(L_n)$. Hence,

$$M_2(Q(L_n)) = 596 + 36(6n - 16) = 308 + 108n$$

and

$$M_2(Q(L_n)) = M_2(R(L_n)) + (548 - 72n)$$

from equation (3.6).

Case 3: If $n \geq 5$, $Q(L_n)$ contains 4 pairs of vertices of degree 4 and 5, four pairs of vertices of degree 4 and 2, four pairs of vertices of degree 2 and 5, eight pairs of vertices of degree 5 and 6, four pairs of vertices of degree 5 and 3 , $6n - 16$ pairs of vertices of degree 3 and 6, $6n - 18$ pairs of vertices of degree 6 are adjacent to each other. Hence,

$$\begin{aligned} M_2(Q(L_n)) &= 452 + 18(6n - 16) + 36(6n - 18) \\ &= 2M_2(R(L_n)) + (-36n - 4) \end{aligned}$$

by equation (3.6). □

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Total Minimal Dominating Signed Graph

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Abstract: A *Smarandachely k-signed graph* (*Smarandachely k-marked graph*) is an ordered pair $S = (G, \sigma)$ ($S = (G, \mu)$) where $G = (V, E)$ is a graph called *underlying graph of S* and $\sigma : E \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k)$ ($\mu : V \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k)$) is a function, where each $\overline{e}_i \in \{+, -\}$. Particularly, a Smarandachely 2-signed graph or Smarandachely 2-marked graph is called abbreviated a *signed graph* or a *marked graph*. In this paper, we define the *total minimal dominating signed graph* $M_t(S) = (M_t(G), \sigma)$ of a given signed digraph $S = (G, \sigma)$ and offer a structural characterization of total minimal dominating signed graphs. Further, we characterize signed graphs S for which $S \sim M_t(S)$ and $L(S) \sim M_t(S)$, where $\sim$ denotes switching equivalence and $M_t(S)$ and $L(S)$ are denotes total minimal dominating signed graph and line signed graph of S respectively.

Key Words: Smarandachely k-signed graphs, Smarandachely k-marked graphs, signed graphs, marked graphs, balance, switching, total minimal dominating signed graph, line signed graphs, negation.

AMS(2000): 05C22

§1. Introduction

Unless mentioned or defined otherwise, for all terminology and notion in graph theory the reader is refer to [8]. We consider only finite, simple graphs free from self-loops.

A *Smarandachely k-signed graph* (*Smarandachely k-marked graph*) is an ordered pair $S = (G, \sigma)$ ($S = (G, \mu)$) where $G = (V, E)$ is a graph called *underlying graph of S* and $\sigma : E \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k)$ ($\mu : V \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k)$) is a function, where each $\overline{e}_i \in \{+, -\}$. Particularly, a Smarandachely 2-signed graph or Smarandachely 2-marked graph is called abbreviated a *signed graph* or a *marked graph*. Cartwright and Harary [5] considered graphs in which vertices represent persons and the edges represent symmetric dyadic relations amongst persons each of which designated as being positive or negative according to whether the nature of the

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relationship is positive (friendly, like, etc.) or negative (hostile, dislike, etc.). Such a network S is called a *signed graph* (Chartrand [6]; Harary et al. [11]).

Signed graphs are much studied in literature because of their extensive use in modeling a variety socio-psychological process (e.g., see Katai and Iwai [13], Roberts [15] and Roberts and Xu [16]) and also because of their interesting connections with many classical mathematical systems (Zaslavsky [22]).

A cycle in a signed graph S is said to be *positive* if the product of signs of its edges is positive. A cycle which is not positive is said to be *negative*. A signed graph is then said to be *balanced* if every cycle in it is positive (Harary [9]). Harary and Kabell [22] developed a simple algorithm to detect balance in signed graphs as also enumerated them.

A *marking* of S is a function $\mu : V(G) \rightarrow \{+, -\}$; A signed graph S together with a marking μ is denoted by S_μ. Given a signed graph S one can easily define a marking μ of S as follows: For any vertex $v \in V(S)$,

$$\mu(v) = \prod_{uv \in E(S)} \sigma(uv),$$

the marking μ of S is called *canonical marking* of S.

The following characterization of balanced signed graphs is well known.

Theorem 1(E. Sampathkumar [17]) *A signed graph $S = (G, \sigma)$ is balanced if, and only if, there exists a marking μ of its vertices such that each edge uv in S satisfies $\sigma(uv) = \mu(u)\mu(v)$.*

The idea of switching a signed graph was introduced by Abelson and Rosenberg [1] in connection with structural analysis of marking μ of a signed graph S. Switching S with respect to a marking μ is the operation of changing the sign of every edge of S to its opposite whenever its end vertices are of opposite signs. The signed graph obtained in this way is denoted by $\mathcal{S}_\mu(S)$ and is called *μ-switched signed graph* or just *switched signed graph*. Two signed graphs $S_1 = (G, \sigma)$ and $S_2 = (G', \sigma')$ are said to be *isomorphic*, written as $S_1 \cong S_2$ if there exists a graph isomorphism $f : G \rightarrow G'$ (that is a bijection $f : V(G) \rightarrow V(G')$ such that if uv is an edge in G then $f(u)f(v)$ is an edge in G') such that for any edge $e \in G$, $\sigma(e) = \sigma'(f(e))$. Further a signed graph $S_1 = (G, \sigma)$ *switches* to a signed graph $S_2 = (G', \sigma')$ (or that S_1 and S_2 are *switching equivalent*) written $S_1 \sim S_2$, whenever there exists a marking μ of S_1 such that $\mathcal{S}_\mu(S_1) \cong S_2$. Note that $S_1 \sim S_2$ implies that $G \cong G'$, since the definition of switching does not involve change of adjacencies in the underlying graphs of the respective signed graphs.

Two signed graphs $S_1 = (G, \sigma)$ and $S_2 = (G', \sigma')$ are said to be *weakly isomorphic* (see [20]) or *cycle isomorphic* (see [21]) if there exists an isomorphism $\phi : G \rightarrow G'$ such that the sign of every cycle Z in S_1 equals to the sign of $\phi(Z)$ in S_2. The following result is well known (See [21]):

Theorem(T. Zaslavsky [21]) *Two signed graphs S_1 and S_2 with the same underlying graph are switching equivalent if, and only if, they are cycle isomorphic.*

§2. Total Minimal Dominating Signed Graph

The total minimal dominating graph $M_t(G)$ of a graph G is the intersection graph on the family of all total minimal dominating sets of vertices in G. This concept was introduced by Kulli and Iyer [14].

We now extend the notion of $M_t(G)$ to the realm of signed graphs. The *total minimal dominating signed graph* $M_t(S)$ of a signed graph $S = (G, \sigma)$ is a signed graph whose underlying graph is $M_t(G)$ and sign of any edge uv is $M_t(S)$ is $\mu(u)\mu(v)$, where μ is the canonical marking of S. Further, a signed graph $S = (G, \sigma)$ is called total minimal dominating signed graph, if $S \cong M_t(S')$ for some signed graph S'. The following result restricts the class of total minimal dominating signed graphs.

Theorem 3 *For any signed graph $S = (G, \sigma)$, its total minimal dominating signed graph $M_t(S)$ is balanced.*

Proof Since sign of any edge uv is $M_t(S)$ is $\mu(u)\mu(v)$, where μ is the canonical marking of S, by Theorem 1, $M_t(S)$ is balanced. □

For any positive integer k, the k^{th} iterated total minimal dominating signed graph, $M_t^k(S)$ of S is defined as follows:

$$M_t^0(S) = S,\ M_t^k(S) = M_t(M_t^{k-1}(S))$$

Corollary 4 *For any signed graph $S = (G, \sigma)$ and for any positive integer k, $M_t^k(S)$ is balanced.*

The following result characterizes signed graphs which are total minimal dominating signed graphs.

Theorem 5 *A signed graph $S = (G, \sigma)$ is a total minimal dominating signed graph if, and only if, S is balanced signed graph and its underlying digraph G is a total minimal dominating graph.*

Proof Suppose that S is total minimal dominating signed graph. Then there exists a signed graph $S' = (G', \sigma')$ such that $M_t(S') \cong S$. Hence by definition $M_t(G) \cong G'$ and by Theorem 3, S is balanced.

Conversely, suppose that $S = (G, \sigma)$ is balanced and G is total minimal dominating graph. That is there exists a graph G' such that $M_t(G') \cong G$. Since S is balanced by Theorem 1, there exists a marking μ of vertices of S such that for any edge $uv \in G$, $\sigma(uv) = \mu(u)\mu(v)$. Also since $G \cong M_t(G')$, vertices in G are in one-to-one correspondence with the edges of G'. Now consider the signed graph $S' = (G', \sigma')$, where for any edge e' in G' to be the marking on the corresponding vertex in G. Then clearly $M_t(S') \cong S$ and so S is total minimal dominating graph. □

In [3], the authors proved the following for a graph G its total minimal dominating graph $M_t(G)$ is isomorphic to G then G is either C_3 or C_4. Analogously we have the following.

Theorem 6 *For any signed graph $S = (G, \sigma)$, $S \sim M_t(S)$ if, and only if, G is isomorphic to either C_3 or C_4 and S is balanced.*

Proof Suppose $S \sim M_t(S)$. This implies, $G \cong M_t(G)$ and hence by the above observation we see that the graph G must be isomorphic to either C_3 or C_4. Now, if S is any signed graph on any one of these graphs, Theorem 3 implies that $M_t(S)$ is balanced and hence if S is unbalanced its $M_t(S)$ being balanced cannot be switching equivalent to S in accordance with Theorem 2. Therefore, S must be balanced.

Conversely, suppose that S is balanced signed graph on C_3 or C_4. Then, since $M_t(S)$ is balanced as per Theorem 3, the result follows from Theorem 2 again. □

Behzad and Chartrand [4] introduced the notion of line signed graph $L(S)$ of a given signed graph S as follows: Given a signed graph $S = (G, \sigma)$ its *line signed graph* $L(S) = (L(G), \sigma')$ is the signed graph whose underlying graph is $L(G)$, the line graph of G, where for any edge e_ie_j in $L(S)$, $\sigma'(e_ie_j)$ is negative if, and only if, both e_i and e_j are adjacent negative edges in S. Another notion of line signed graph introduced in [7], is as follows: The *line signed graph* of a signed graph $S = (G, \sigma)$ is a signed graph $L(S) = (L(G), \sigma')$, where for any edge ee' in $L(S)$, $\sigma'(ee') = \sigma(e)\sigma(e')$. In this paper, we follow the notion of line signed graph defined by M. K. Gill [7] (See also E. Sampathkumar et al. [18,19]).

Theorem 7(M. Acharya [2]) *For any signed graph $S = (G, \sigma)$, its line signed graph $L(S) = (L(G), \sigma')$ is balanced.*

We now characterize signed graphs whose total minimal dominating signed graphs and its line signed graphs are switching equivalent. In the case of graphs the following result is due to Kulli and Iyer [14].

Theorem 8(Kulli and Iyer [14]) *If G is a $(p-2)$-regular graph then, $M_t(G) \cong L(G)$.*

Theorem 9 *For any signed graph $S = (G, \sigma)$, $M_t(S) \sim L(S)$, if, and only if, G is $(p-2)$-regular.*

Proof Suppose $M_t(S) \sim L(S)$. This implies, $M_t(G) \cong L(G)$ and hence by Theorem 8, we see that the graph G must be $(p-2)$-regular.

Conversely, suppose that G is $(p-2)$-regular. Then $M_t(G) \cong L(G)$ by Theorem 8. Now if S is signed graph with $(p-2)$-regular, then by Theorem 3 and Theorem 7, $M_t(S)$ and $L(S)$ are balanced and hence, the result follows from Theorem 2. □

The notion of *negation* $\eta(S)$ of a given signed graph S defined in [10] as follows:

$\eta(S)$ has the same underlying graph as that of S with the sign of each edge opposite to that given to it in S. However, this definition does not say anything about what to do with nonadjacent pairs of vertices in S while applying the unary operator $\eta(.)$ of taking the negation of S.

Theorem 6 provides easy solutions to two other signed graph switching equivalence relations, which are given in the following results.

Corollary 10 *For any signed graph $S = (G, \sigma)$, $M_t(\eta(S)) \sim M_t(S)$.*

Corollary 11 *For any signed graph $S = (G, \sigma)$, $\eta(S) \sim M_t(S)$ if, and only if, S is an unbalanced signed graph and $G = C_3$.*

For a signed graph $S = (G, \sigma)$, the $M_t(S)$ is balanced (Theorem 3). We now examine, the conditions under which negation $\eta(S)$ of $M_t(S)$ is balanced.

Corollary 12 *Let $S = (G, \sigma)$ be a signed graph. If $M_t(G)$ is bipartite then $\eta(M_t(S))$ is balanced.*

Proof Since, by Theorem 3 $M_t(S)$ is balanced, if each cycle C in $M_t(S)$ contains even number of negative edges. Also, since $M_t(G)$ is bipartite, all cycles have even length; thus, the number of positive edges on any cycle C in $M_t(S)$ is also even. Hence $\eta(M_t(S))$ is balanced.□

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The Number of Minimum Dominating Sets in $P_n \times P_2$

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Abstract: A set S of vertices in a graph G is said to be a *Smarandachely k-dominating set* if each vertex of G is dominated by at least k vertices of S. The *Smarandachely k-domination number* $\gamma_k(G)$ of G is the minimum cardinality of Smarandachely k-dominating sets of G. Particularly, if $k = 1$, a Smarandachely k-dominating set is called a *dominating set* of G and $\gamma_k(G)$ is abbreviated to $\gamma(G)$. In this paper, we get the Smarandachely 1-dominating number, i.e., the dominating number of $P_n \times P_2$.

Key Words: Smarandachely k-dominating set, Smarandachely k-domination number, dominating sets, dominating number.

AMS(2000): 05C69

§1. Introduction

We considered finite, undirected, simple graphs $G = (V, E)$ with vertex set $V(G)$ and edge set $E(G)$. The order of G is given by $n = |V(G)|$. A set $S \subseteq V$ of vertices in a graph G is called a dominating set if every vertex $v \in V$ is either an element of S or is adjacent to an element of S. A dominating set S is a minimum dominating set if no proper subset is a dominating set. The domination number $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set in G. A set of vertices S in a graph G is said to be a *Smarandachely k-dominating set* if each vertex of G is dominated by at least k vertices of S. Particularly, if $k = 1$, such a set is called a dominating set of G. The *Smarandachely k-domination number* $\gamma_k(G)$ of G is the minimum cardinality of a Smarandachely k-dominating set of G.

As known, a fundamental unsolved problem concerning the bounds on the domination number of product graphs is to settle Vizing's conjecture. Another basic problem is to find the domination number or bound on the domination number of specific Cartesian products, for example the $j \times k$ grid graph $P_j \times P_k$. This too seems to be a difficult problem. It is

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known that dominating set remains NP- complete when restricted to arbitrary sub graphs of [2,12]. However, Hare, Hare and Hedetniemi [8,9] developed a linear time algorithm to solve this problem on $j \times k$ grid graph for any fixed j. Moreover, the domination number of $P_j \times P_k$ has been determined for small values of j. Jacobson and Kinch [10] established it for $j = 1, 2, 3, 4$ and all k. Hare [8] developed algorithm which she used to conjecture simple formulae for $\gamma(P_j \times P_k)$ for $1 \leq j \leq 10$. Chang and Clark [4] proved Hare's formulae for the domination number of $P_5 \times P_k$ and $P_6 \times P_k$. The domination numbers for $P_j \times P_k$ $1 \leq j \leq 6$ are listed below:

1. $\gamma(P_1 \times P_k) = \lfloor \frac{k+2}{3} \rfloor$, $k \geq 1$
2. $\gamma(P_2 \times P_k) = \lfloor \frac{k+2}{2} \rfloor$, $k \geq 1$
3. $\gamma(P_3 \times P_k) = \lfloor \frac{3k+4}{4} \rfloor$, $k \geq 1$
4. $\gamma(P_3 \times P_k) = \begin{cases} k+1, & k = 1, 2, 3, 5, 6, 9; \\ k, & \text{otherwise.} \end{cases}$
5. $\gamma(P_3 \times P_k) = \begin{cases} \frac{6k+6}{5}, & k = 2, 3, 7; \\ \frac{6k+8}{5}, & \text{otherwise.} \end{cases}$
6. $\gamma(P_3 \times P_k) = \begin{cases} \frac{10k+10}{7}, & k \geq 6k \equiv 1 mod 7; \\ \frac{10k+12}{7}, & \text{otherwise} if k \geq 4. \end{cases}$

It is well known that the concept of domination is originated from the game of chess board. The problem of finding the minimum number of stones is one aspect and the number of ways of placing the minimum number of stones is another aspect. Though the first aspect has not been resolved as mentioned earlier, we consider the second aspect of the problem, that is, finding the number of ways of placing the minimum number of stones. In this paper, we consider the second aspect of the problem for $P_n \times P_2$. That is, equivalently finding the minimum number of dominating sets in $P_n \times P_2$.

$P_7 \times P_2$:

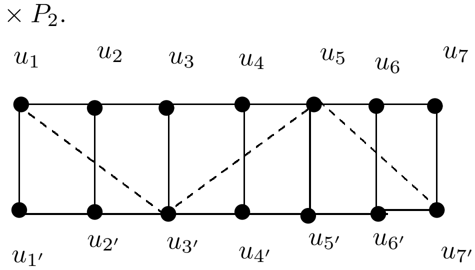

Figure 1: $P_7 \times P_2$ with dominating vertices

The minimum dominating sets of Figure 1 are $\{u_1, u_{3'}, u_5, u_{7'}\}$ and $\{u_{1'}, u_3, u_{5'}, u_7\}$.

Similarly, the minimum dominating sets of Figure 2 are: $\{u_1, u_{3'}, u_5, u_{6'}\}$ $\{u_{1'}, u_3, u_{5'}, u_6\}$, $\{u_1, u_{3'}, u_5, u_6\}$, $\{u_{1'}, u_3, u_{5'}, u_{6'}\}$, $\{u_1, u_{3'}, u_4, u_{6'}\}$, $\{u_{1'}, u_3, u_{4'}, u_6\}$, $\{u_1, u_{2'}, u_4, u_{6'}\}$, $\{u_{1'}, u_2,$

$P_6 \times P_2$:

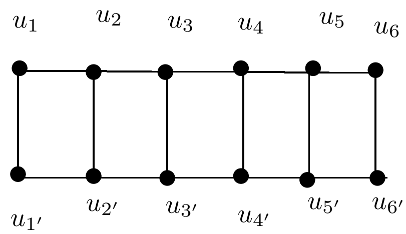

Figure 2: $P_6 \times P_2$ with dominating vertices

$u_{4'}, u_6\}$, $\{u_{1'}, u_3, u_4, u_{6'}\}$, $\{u_1, u_{3'}, u_{4'}, u_6\}$, $\{u_1, u_2, u_{4'}, u_6\}$, $\{u_2, u_{2'}, u_4, u_{6'}\}$, $\{u_{2'}, u_2, u_{4'}, u_6\}$, $\{u_{1'}, u_3, u_{5'}, u_5\}$, $\{u_1, u_{3'}, u_{5'}, u_5\}$, $\{u_2, u_{2'}, u_5, u_{5'}\}$, $\{u_{1'}, u_{2'}, u_4, u_{6'}\}$.

As such the domination number of $P_n \times P_2$ is, $\gamma(P_n \times P_2) = \left\lfloor \frac{n+2}{2} \right\rfloor$. Using this value we consider the minimum number of dominating sets $\gamma_D(P_n \times P_2)$ for the values $n = 2k+1$ and $n = 2k$.

§2. Results

To prove our results, we need some lemmas proved below.

Lemma 2.1 *Let vertices of first and second rows in $P_{2k+1} \times P_2$ are labeled with $v_1, v_2 \ldots, v_{2k-2}$, $v_{2k-1}, v_{2k}, v_{2k+1}$ and $u_1, u_2, \ldots, u_{2k-2}, u_{2k-1}, u_{2k}, u_{2k+1}$, then there is no md-set containing both the vertices v_{2k} and u_{2k}.*

Proof On the contrary, assume that there is an md-set say D in $P_{2k+1} \times P_2$ containing both the vertices v_{2k} and u_{2k}. Clearly, $D - \{v_{2k}u_{2k}\}$ dominating set in $P_{2k-2} \times P_2$, for otherwise there exists a vertex v_i (or u_i) of $P_{2k-2} \times P_2$ which is not either in $D - \{v_{2k}u_{2k}\}$ or not adjacent to any vertex of $D - \{v_{2k}u_{2k}\}$ then this vertex v_i (or u_i) is not in D or is not adjacent to any vertex of D in $P_{2k+1} \times P_2$ and hence D is not a dominating set in $P_{2k+1} \times P_2$, a contradiction to the assumption.

Therefore,$K = \gamma(P_{2k-2} \times P_2) \leq |D - \{v_{2k}u_{2k}\}| = |D| - 2 = k + 1 - 2 = k - 1$ a contradiction, which proves the Lemma. □

Lemma 2.2 *There is no md-set containing both v_{2k+1} and u_{2k+1} , where the vertices of $P_{2k+1} \times P_2$ are labelled as in the above Lemma 2.1.*

Proof The proof is similar to that of Lemma 2.1 with a slight change, that is by considering $D - \{v_{2k+1}u_{2k+1}\}$ which is the dominating set in $P_{2k-1} \times P_2$ with D being a md - set containing both v_{2k+1} and u_{2k+1} in $P_{2k-1} \times P_2$. Thus, $K = \gamma(P_{2k-1} \times P_2) \leq |D - \{v_{2k+1}u_{2k+1}\}| = |D| - 2 = k + 1 - 2 = k - 1$ a contradiction, which proves that D is not an md - set. □

Corollary 2.3 *Every md - set in $P_{2k+1} \times P_2$ contains either v_{2k+1} or u_{2k+1}.*

Theorem 2.4 $\gamma(P_{2k+1} \times P_2) = \begin{cases} 3, & if\ k = 1; \\ 2, & if\ k \geq 2. \end{cases}$

Lemma 2.5 *There exists exactly two md - sets containing both v_{2k-1} and u_{2k-1} in $P_{2k} \times P_2$.*

Proof In $P_{2k} \times P_2$, clearly the vertices v_{2k-1} and u_{2k-1} can cover v_{2k-2}, v_{2k} and u_{2k-2}, u_{2k} respectively. We claim that any md - set D containing either v_{2k-3} or u_{2k-3} but not both, (follows from the Corollary 2.3)union $\{v_{2k-1}, u_{2k-1}\}$ is an md - set in $P_{2k} \times P_2$. Since $k + 1 = \gamma(P_{2k} \times P_2) \leq |D \cup \{v_{2k-1}, v_{2k-2}\}| = \gamma(P_{2k-3} \times P_2) + 2 = k - 1 + 2 = k + 1$. Hence the claim. Again by Theorem 2.4 and Corollary 2.3, there are exactly two md-sets viz D_1 containing v_{2k-3} and D_2 containing u_{2k-3} in $P_{2k-3} \times P_2$. Hence $D_1 \cup \{v_{2k-1}, u_{2k-1}\}$ and $D_2 \cup \{v_{2k-1}, u_{2k-1}\}$ are md-sets in $P_{2k} \times P_2$. □

Lemma 2.6 *There is no md-set containing both v_{2k} and u_{2k} in $P_{2k} \times P_2$.*

Proof On the contrary, assume that there is a md - set in $P_{2k} \times P_2$ containing both v_{2k} and u_{2k}. Then, clearly,

$D - \{v_{2k}, u_{2k}\}$ is a dominating set in $P_{2k} \times P_2$. Thus, $k = \gamma(P_{2k-2} \times P_2) \leq |D - \{v_{2k}, u_{2k}\}| \leq |D| - 2 = k + 1 - 2 = k - 1$ a contradiction, which proves this lemma. □

Theorem 2.7 *For any $k \geq 3$, $\gamma_D\,(P_{2k} \times P_2) = \gamma(P_{2k-2} \times P_2) + 4$*

Proof We prove this theorem by four steps following.

Step 1. Let $D_1, D_2, \cdots, D_t$ be md-sets containing u_{2k-2} in $P_{2k-2} \times P_2$, then, $D_i \cup \{u_{2k}\}$ and $D_i \cup \{v_{2k}\}$ are dominating sets in $P_{2k-2} \times P_2$ for $i = 1, 2, \cdots, t$ But, $k + 1 = \gamma(P_{2k} \times P_2) \leq |D_i| \cup \{u_{2k}\} = |D_i| + 1 = \gamma(P_{2k-2} \times P_2) + 1 = k + 1$. Hence, $D_i \cup \{u_{2k}\}$ is a md-set in $P_{2k} \times P_2$. And for the same reason, $D_i \cup \{v_{2k}\}$ is a md-set in $P_{2k} \times P_2$.

Step 2. By the Lemma 2.5, Let D_1 and D_2 be two md - sets containing both v_{2k-3} and u_{2k-3} in $P_{2k-2} \times P_2$. But, by the Lemma, there exists exactly two md - sets say D_1' and D_2' containing v_{2k-3} and u_{2k-3} respectively in $P_{2k-2} \times P_2$. So, D_1 must be obtained from $D_1' \cup \{v_{2k-3}, u_{2k-3}\}$ and D_2 must be obtained from $D_2' \cup \{v_{2k-3}, u_{2k-3}\}$. Thus it is not difficult to see that $(D_1 - v_{2k-3}) \cup \{v_{2k-1}, u_{2k}\}$ and $(D_1 - u_{2k-3}) \cup \{u_{2k-1}, v_{2k}\}$ are md- sets in $P_{2k} \times P_2$.

Step 3. For md-sets D_1 and D_2 of $P_{2k-2} \times P_2$ the sets $(D_1 - \{v_{2k-3}\}) \cup \{v_{2k-1}, v_{2k}\}$ and $(\{D_2 - u_{2k-3}\}) \cup \{u_{2k-1}, u_{2k},\}$ are md- sets in $P_{2k} \times P_2$.

Step 4. For md-sets D_1 and D_2 of $P_{2k-2} \times P_2$ the sets $(D_1 - \{v_{2k-3}\}) \cup \{v_{2k-1}, u_{2k-1}\}$ and $(\{D_2 - u_{2k-3}\}) \cup \{v_{2k-1}, u_{2k-1}\}$ are md- sets in $P_{2k} \times P_2$.

Thus $\gamma_D\,(P_{2k} \times P_2) = 2t + 2 + 2 + 2 = 2t + 2 + 4 = \gamma_D\,(P_{2k-2} \times P_2) + 4$ by steps $1, 2, 3, 4$.□

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Super Fibonacci Graceful Labeling

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Abstract: A *Smarandache-Fibonacci Triple* is a sequence $S(n),\ n \geq 0$ such that $S(n) = S(n-1)+S(n-2)$, where $S(n)$ is the Smarandache function for integers $n \geq 0$. Certainly, it is a generalization of *Fibonacci sequence*. A *Fibonacci graceful labeling* and a *super Fibonacci graceful labeling* on graphs were introduced by Kathiresan and Amutha in 2006. Generally, let G be a (p,q)-graph and $\{S(n)|n \geq 0\}$ a Smarandache-Fibonacci Triple. An bijection $f\colon V(G) \to \{S(0), S(1), S(2), \ldots, S(q)\}$ is said to be a *super Smarandache-Fibonacci graceful graph* if the induced edge labeling $f^*(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{S(1), S(2), \ldots, S(q)\}$. Particularly, if $S(n), n \geq 0$ is just the Fibonacci sequence $F_i,\ i \geq 0$, such a graph is called a *super Fibonacci graceful graph*. In this paper, we construct new types of graphs namely $F_n \oplus K^+_{1,m}$, $C_n \oplus P_m$, $K_{1,n} \oslash K_{1,2}$, $F_n \oplus P_m$ and $C_n \oplus K_{1,m}$ and we prove that these graphs are super Fibonacci graceful graphs.

Key Words: Smarandache-Fibonacci triple, graceful labeling, Fibonacci graceful labeling, super Smarandache-Fibonacci graceful graph, super Fibonacci graceful graph.

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§1. Introduction

By a graph, we mean a finite undirected graph without loops or multiple edges. A path of length n is denoted by P_{n+1}. A cycle of length n is denoted by C_n. G^+ is a graph obtained from the graph G by attaching pendant vertex to each vertex of G. Graph labelings, where the vertices are assigned certain values subject to some conditions, have often motivated by practical problems.

In the last five decades enormous work has been done on this subject [1]. The concept of graceful labeling was first introduced by Rosa [5] in 1967. A function f is a graceful labeling of a graph G with q edges if f is an injection from the vertices of G to the set $\{0, 1, 2, \ldots, q\}$ such that when each edge uv is assigned the label $|f(u)-f(v)|$, the resulting edge labels are distinct. The notion of Fibonacci graceful labeling and Super Fibonacci graceful labeling were introduced by Kathiresan and Amutha [3]. We call a function f, a *Fibonacci graceful labeling* of a graph G

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with q edges if f is an injection from the vertices of G to the set $\{0, 1, 2, \ldots, F_q\}$, where F_q is the q^{th} Fibonacci number of the Fibonacci series $F_1 = 1, F_2 = 2, F_3 = 3$, $F_4 = 5, \cdots$, such that each edge uv is assigned the labels $|f(u) - f(v)|$, the resulting edge labels are $F_1, F_2, \ldots, F_q$. An injective function $f : V(G) \to \{F_0, F_1, \ldots, F_q\}$, where F_q is the q^{th} Fibonacci number, is said to be a super Fibonacci graceful labeling if the induced edge labeling $|f(u) - f(v)|$ is a bijection onto the set $\{F_1, F_2, \ldots, F_q\}$. In the labeling problems the induced labelings must be distinct. So to introduce Fibonacci graceful labelings we assume $F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, \cdots$, as the sequence of Fibonacci numbers instead of $0, 1, 2, \ldots$, [3].

Generally, a *Smarandache-Fibonacci Triple* is a sequence $S(n)$, $n \geq 0$ such that $S(n) = S(n-1) + S(n-2)$, where $S(n)$ is the Smarandache function for integers $n \geq 0$ [2]. A (p, q)-graph G is a *super Smarandache-Fibonacci graceful graph* if there is an bijection $f\colon V(G) \to \{S(0), S(1), S(2), \ldots, S(q)\}$ such that the induced edge labeling $f^*(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{S(1), S(2), \ldots, S(q)\}$. So a super Fibonacci graceful graph is a special type of Smarandache-Fibonacci graceful graph by definition.

In this paper, we prove that $F_n \oplus K_{1,m}^{+}$, $C_n \oplus P_m$, $K_{1,n} \oslash K_{1,2}$, $F_n \oplus P_m$ and $C_n \oplus K_{1,m}$ are super Fibonacci graceful graphs.

§2. Main Results

In this section, we show that some new types of graphs namely $F_n \oplus K_{1,m}^{+}$, $C_n \oplus P_m$, $K_{1,n} \oslash K_{1,2}$, $F_n \oplus P_m$ and $C_n \oplus K_{1,m}$ are super Fibonacci graceful graphs.

Definition 2.1([4]) *Let G be a (p, q) graph. An injective function $f\colon V(G) \to \{F_0, F_1, F_2, \ldots, F_q\}$, where F_q is the q^{th} Fibonacci number, is said to be a super Fibonacci graceful graphs if the induced edge labeling $f^*(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{F_1, F_2, \ldots, F_q\}$.*

Definition 2.2 *The graph $G = F_n \oplus P_m$ consists of a fan F_n and a Path P_m which is attached with the maximum degree of the vertex of F_n.*

The following theorem shows that the graph $F_n \oplus P_m$ is a super Fibonacci graceful graph.

Theorem 2.3 *The graph $G = F_n \oplus P_m$ is a super Fibonacci graceful graph.*

Proof Let $\{u_0 = v, u_1, u_2, \ldots, u_n\}$ be the vertex set of F_n and $v_1, v_2, \ldots, v_m$ be the vertices of P_m joined with the maximum degree of the vertex u_0 of F_n. Also, $|V(G)| = m + n + 1$ and $|E(G)| = 2n + m - 1$. Define $f : V(G) \to \{F_0, F_1, \ldots, F_q\}$ by $f(u_0) = F_0$, $f(u_i) = F_{2n+m-1-2(i-1)}$, $1 \leq i \leq n$, $f(v_i) = F_{m-2(i-1)}$, $1 \leq i \leq 2$,

$$f(v_m) = \begin{cases} F_2 & \text{if} m \equiv 0(\text{mod} 3) \\ F_1 & \text{if} m \equiv 1, 2(\text{mod} 3) \end{cases} \qquad f(v_{m-1}) = \begin{cases} F_3 & \text{if} m \equiv 1(\text{mod} 3) \\ F_2 & \text{if} m \equiv 2(\text{mod} 3) \end{cases}$$

and $f(v_{m-2}) = F_4$ if $m \equiv 2(mod 3)$.

For $l = 1, 2, \ldots, \dfrac{m-3}{3}$, or $\dfrac{m-4}{3}$, or $\dfrac{m-5}{3}$ according to $m \equiv 0(mod 3)$ or $m \equiv 1(mod 3)$

or $m \equiv 2(mod3)$, define

$$f(v_{i+2}) = F_{m-1-2(i-1)+3(l-1)}, \quad 3l-2 \leq i \leq 3l.$$

We claim that the edge labels are distinct. Let $E_1 = \{f^*(u_iu_{i+1}) : i = 1, 2, \ldots, n-1\}$. Then

$$\begin{aligned} E_1 &= \{|f(u_i) - f(u_{i+1})| : i = 1, 2, \ldots, n-1\} \\ &= \{|f(u_1) - f(u_2)|, |f(u_2) - f(u_3)|, \ldots, |f(u_{n-1}) - f(u_n)|\} \\ &= \{|F_{2n+m-1} - F_{2n+m-3}|, |F_{2n+m-3} - F_{2n+m-5}|, \ldots, |F_{m+3} - F_{m+1}|\} \\ &= \{F_{2n+m-2}, F_{2n+m-4}, \ldots, F_{m+4}, F_{m+2}\}, \end{aligned}$$

$$\begin{aligned} E_2 &= \{f^*(u_0u_i) : i = 1, 2, \ldots, n\} = \{|f(u_0) - f(u_i)| : i = 1, 2, \ldots, n\} \\ &= \{|f(u_0) - f(u_1)|, |f(u_0) - f(u_2)|, \ldots, |f(u_0) - f(u_n)|\} \\ &= \{|F_0 - F_{2n+m-1}|, |F_0 - F_{2n+m-3}|, \ldots, |F_0 - F_{m+1}|\} \\ &= \{F_{2n+m-1}, F_{2n+m-3}, \ldots, F_{m+3}, F_{m+1}\}, \end{aligned}$$

$$\begin{aligned} E_3 &= \{f^*(u_0v_1), f^*(v_1v_2)\} = \{|f(u_0) - f(v_1)|, |f(v_1) - f(v_2)|\} \\ &= \{|F_0 - F_m|, |F_m - F_{m-2}|\} = \{F_m, F_{m-1}\}. \end{aligned}$$

Let $E_4 = \{f^*(v_2v_3)\}$. The edge labeling between the vertex v_2 and starting vertex v_3 of the first loop is

$$E_4 = \{|f(v_2) - f(v_3)|\} = \{|F_{m-2} - F_{m-1}|\} = \{F_{m-3}\}.$$

For $l = 1$, let $E_5 = \{f^*(v_{i+2}v_{i+3}) : 1 \leq i \leq 2\}$. Then

$$\begin{aligned} E_5 &= \{|f(v_{i+2}) - f(v_{i+3})| : 1 \leq i \leq 2\} \\ &= \{|f(v_3) - f(v_4)|, |f(v_4) - f(v_5)|\} \\ &= \{|F_{m-1} - F_{m-3}|, |F_{m-3} - F_{m-5}|\} = \{F_{m-2}, F_{m-4}\}. \end{aligned}$$

Let $E_5^1 = \{f^*(v_5v_6)\}$. We find the edge labeling between the end vertex v_5 of the first loop and starting vertex v_6 of the second loop following.

$$E_5^1 = \{|f(v_5) - f(v_6)|\} = \{|F_{m-5} - F_{m-4}|\} = \{F_{m-6}\}.$$

For $l = 2$, let $E_6 = \{f^*(v_{i+2}v_{i+3}) : 4 \leq i \leq 5\}$. Then

$$\begin{aligned} E_6 &= \{|f(v_{i+2}) - f(v_{i+3})| : 4 \leq i \leq 5\} = \{|f(v_6) - f(v_7)|, |f(v_7) - f(v_8)|\} \\ &= \{|F_{m-4} - F_{m-6}|, |F_{m-6} - F_{m-8}|\} = \{F_{m-5}, F_{m-7}\}. \end{aligned}$$

For labeling between the end vertex v_8 of the second loop and starting vertex v_9 of the third loop, let $E_6^1 = \{f^*(v_8v_9)\}$. Then

$$E_6^1 = \{|f(v_8) - f(v_9)|\} = \{|F_{m-8} - F_{m-7}|\} = \{F_{m-9}\},$$

etc.. For $l = \dfrac{m-5}{3} - 1$, let $E_{\frac{m-5}{3}-1} = \{f^*(v_{i+2}v_{i+3}) : m-10 \leq i \leq m-9\}$. Then

$$\begin{aligned} E_{\frac{m-5}{3}-1} &= \{|f(v_{i+2}) - f(v_{i+3})| : m-10 \leq i \leq m-9\} \\ &= \{|f(v_{m-8}) - f(v_{m-7})|, |f(v_{m-7}) - f(v_{m-6})|\} \\ &= \{|F_{10} - F_8|, |F_8 - F_6|\} = \{F_9, F_7\}. \end{aligned}$$

For the edge labeling between the end vertex v_{m-6} of the $(\dfrac{m-5}{3} - 1)^{th}$ loop and starting vertex v_{m-5} of the $(\dfrac{m-5}{3})^{rd}$ loop, let $E^1_{\frac{m-5}{3}-1} = \{f^*(v_{m-6}v_{m-5})\}$. Then

$$E^1_{\frac{m-5}{3}-1} = \{|f(v_{m-6}) - f(v_{m-5})|\} = \{|F_6 - F_7|\} = \{F_5\},$$

$$\begin{aligned} E_{\frac{m-5}{3}} &= \{f^*(v_{i+2}v_{i+3}) : m-7 \leq i \leq m-6\} \\ &= \{|f(v_{i+2}) - f(v_{i+3})| : m-7 \leq i \leq m-6\} \\ &= \{|f(v_{m-5}) - f(v_{m-4})|, |f(v_{m-4}) - f(v_{m-3})|\} \\ &= \{|F_7 - F_5|, |F_5 - F_3|\} = \{F_6, F_4\}. \end{aligned}$$

For $l = \dfrac{m-4}{3} - 1$, let $E_{\frac{m-4}{3}-1} = \{f^*(v_{i+2}v_{i+3}) : m-9 \leq i \leq m-8\}$. Then

$$\begin{aligned} E_{\frac{m-4}{3}-1} &= \{|f(v_{i+2}) - f(v_{i+3})| : m-9 \leq i \leq m-8\} \\ &= \{|f(v_{m-7}) - f(v_{m-6})|, |f(v_{m-6}) - f(v_{m-5})|\} \\ &= \{|F_9 - F_7|, |F_7 - F_5|\} = \{F_8, F_6\}. \end{aligned}$$

For the edge labeling between the end vertex v_{m-5} of the $(\dfrac{m-4}{3} - 1)^{th}$ loop and starting vertex v_{m-4} of the $(\dfrac{m-4}{3})^{rd}$ loop, let $E^1_{\frac{m-4}{3}-1} = \{f^*(v_{m-5}v_{m-4})\}$. Then

$$E^1_{\frac{m-4}{3}-1} = \{|f(v_{m-5}) - f(v_{m-4})|\} = \{|F_5 - F_6|\} = \{F_4\}.$$

For $l = \dfrac{m-4}{3}$, let $E_{\frac{m-4}{3}} = \{f^*(v_{i+2}v_{i+3}) : m-6 \leq i \leq m-5\}$. Calculation shows that

$$\begin{aligned} E_{\frac{m-4}{3}} &= \{|f(v_{i+2}) - f(v_{i+3})| : m-6 \leq i \leq m-5\} \\ &= \{|f(v_{m-4}) - f(v_{m-3})|, |f(v_{m-3}) - f(v_{m-2})|\} \\ &= \{|F_6 - F_4|, |F_4 - F_2|\} = \{F_5, F_3\}. \end{aligned}$$

Now for $l = \dfrac{m-3}{3} - 1$, let $E_{\frac{m-3}{3}-1} = \{f^*(v_{i+2}v_{i+3}) : m-8 \leq i \leq m-7\}$. Then

$$\begin{aligned} E_{\frac{m-3}{3}-1} &= \{|f(v_{i+2}) - f(v_{i+3})| : m-8 \leq i \leq m-7\} \\ &= \{|f(v_{m-6}) - f(v_{m-5})|, |f(v_{m-5}) - f(v_{m-4})|\} \\ &= \{|F_8 - F_6|, |F_6 - F_4|\} = \{F_7, F_5\}. \end{aligned}$$

Similarly, for finding the edge labeling between the end vertex v_{m-4} of the $(\dfrac{m-3}{3} - 1)^{th}$ loop and starting vertex v_{m-3} of the $(\dfrac{m-3}{3})^{rd}$ loop, let $E^1_{\frac{m-3}{3}-1} = \{f^*(v_{m-4}v_{m-3})\}$. Then

$$E^1_{\frac{m-3}{3}-1} = \{|f(v_{m-4}) - f(v_{m-3})|\} = \{|F_4 - F_2|\} = \{F_3\}.$$

For $l = \dfrac{m-3}{3}$, let $E_{\frac{m-3}{3}} = \{f^*(v_{i+2}v_{i+3}) : m-5 \leq i \leq m-4\}$. Then

$$\begin{aligned} E_{\frac{m-3}{3}} &= \{|f(v_{i+2}) - f(v_{i+3})| : m-5 \leq i \leq m-4\} \\ &= \{|f(v_{m-3}) - f(v_{m-2})|, |f(v_{m-2}) - f(v_{m-1})|\} \\ &= \{|F_5 - F_3|, |F_3 - F_1|\} = \{F_4, F_2\}. \end{aligned}$$

Now let

$$E^{(1)} = \left(E_1 \bigcup E_2 \bigcup, \ldots, \bigcup E_{\frac{m-3}{3}}\right) \bigcup \left(E_5^1 \bigcup E_6^1 \bigcup, \ldots, \bigcup E^1_{\frac{m-3}{3}-1}\right),$$

$$E^{(2)} = \left(E_1 \bigcup E_2 \bigcup, \ldots, \bigcup E_{\frac{m-4}{3}}\right) \bigcup \left(E_5^1 \bigcup E_6^1 \bigcup, \ldots, \bigcup E^1_{\frac{m-4}{3}-1}\right),$$

$$E^{(3)} = \left(E_1 \bigcup E_2 \bigcup, \ldots, \bigcup E_{\frac{m-5}{3}}\right) \bigcup \left(E_5^1 \bigcup E_6^1 \bigcup, \ldots, \bigcup E^1_{\frac{m-5}{3}-1}\right).$$

If $m \equiv 0(mod3)$, let $E_1^* = \{f^*(v_{m-1}v_m)\}$, then $E_1^* = \{|f(v_{m-1} - f(v_m)|\} = \{|F_1 - F_2|\} = \{F_1\}$. Thus,

$$E = E_1^* \bigcup E^{(1)} = \{F_1, F_2, \ldots, F_{2n+m-1}\}.$$

For example the super Fibonacci graceful labeling of $F_4 \oplus P_6$ is shown in Fig.1.

$F_4 \oplus P_6$:

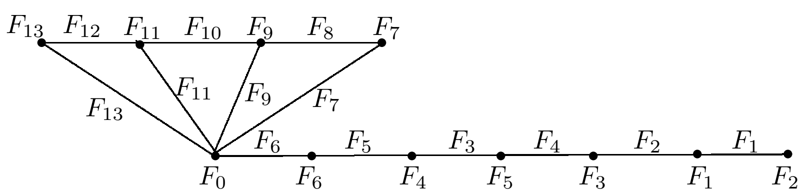

Fig.1

If $m \equiv 1(mod3)$, let $E_2^* = \{f^*(v_{m-2}v_{m-1}), f^*(v_{m-1}v_m)\}$, then

$$\begin{aligned} E_2^* &= \{|f(v_{m-2} - f(v_{m-1}|, |f(v_{m-1} - f(v_m)|\} \\ &= \{|F_2 - F_3|\, |F_3 - F_1|\} = \{F_1, F_2\}. \end{aligned}$$

Thus,

$$E = E_2^* \cup E^{(2)} = \{F_1, F_2, \ldots, F_{2n+m-1}\}.$$

For example the super Fibonacci graceful labeling of $F_4 \oplus P_7$ is shown in Fig.2.

$F_4 \oplus P_7$:

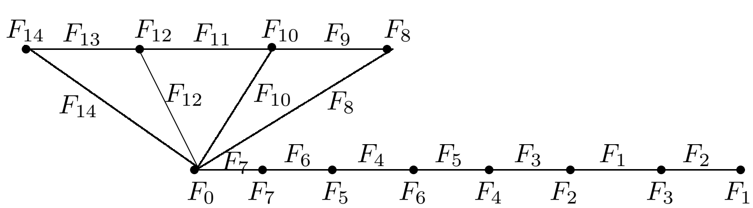

Fig.2

If $m \equiv 2(mod3)$, let $E_3^* = \{f^*(v_{m-3}v_{m-2}), f^*(v_{m-2}v_{m-1}), f^*(v_{m-1}v_m)\}$, then

$$\begin{aligned} E_3^* &= \{|f(v_{m-3}) - f(v_{m-2})|, |f(v_{m-2}) - f(v_{m-1})|, |f(v_{m-1} - f(v_m)|\} \\ &= \{|F_3 - F_4|, |F_4 - F_2|, |F_2 - F_1|\} = \{F_2, F_3, F_1\}. \end{aligned}$$

Thus,

$$E = E_3^* \cup E^{(3)} = \{F_1, F_2, \ldots, F_{2n+m-1}\}.$$

For example the super Fibonacci graceful labeling of $F_5 \oplus P_5$ is shown in Fig.3.

$F_5 \oplus P_5$:

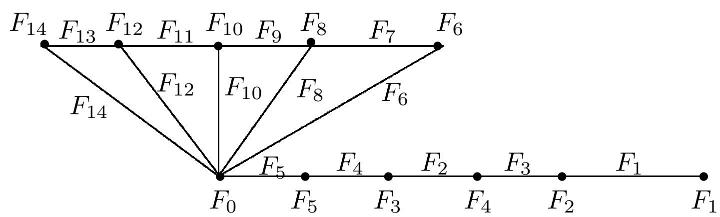

Fig.3

Therefore, $F_n \oplus P_m$ admits a super Fibonacci graceful labeling. Hence, $F_n \oplus P_m$ is a super Fibonacci graceful graph. □

Definition 2.4 *An (n, m)-kite consists of a cycle of length n with m-edge path attached to one vertex and it is denoted by $C_n \oplus P_m$.*

Theorem 2.5 *The graph $G = C_n \oplus P_m$ is a super Fibonacci graceful graph when $n \equiv 0(mod3)$.*

Proof Let $\{u_1, u_2, \ldots, u_n = v\}$ be the vertex set of C_n and $\{v = u_n, v_1, v_2, \ldots, v_m\}$ be the vertex set of P_m joined with the vertex u_n of C_n. Also, $|V(G)| = |E(G)| = m + n$. Define $f : V(G) \to \{F_0, F_1, \ldots, F_q\}$ by $f(u_n) = F_0$, $f(u_1) = F_{m+n}$, $f(u_2) = F_{m+n-2}$ and for $l = 1, 2, \ldots, \dfrac{n-3}{3}$, $f(u_{i+2}) = F_{m+n-1-2(i-1)+3(l-1)}$, and for $3l-2 \le i \le 3l$, $f(v_i) = F_{m-2(i-1)}$, and for $1 \le i \le 2$,

$$f(v_m) = \begin{cases} F_2 & \text{if m} \equiv 0(\text{mod}3) \\ F_1 & \text{if m} \equiv 1, 2(\text{mod}3), \end{cases} \qquad f(v_{m-1}) = \begin{cases} F_3 & \text{if m} \equiv 1(\text{mod}3) \\ F_2 & \text{if m} \equiv 2(\text{mod}3) \end{cases}$$

and $f(v_{m-2}) = F_4$ when $m \equiv 2(mod3)$. For $l = 1, 2, \ldots, \dfrac{m-3}{3}$ or $\dfrac{m-4}{3}$ or $\dfrac{m-5}{3}$ according to $m \equiv 0(mod3)$ or $m \equiv 1(mod3)$ or $m \equiv 2(mod3)$, let $f(v_{i+2}) = F_{m-1-2(i-1)+3(l-1)}$ for $3l - 2 \le i \le 3l$.

We claim that all these edge labels are distinct. Let $E_1 = \{f^* u_n u_1), f^*(u_1 u_2)\}$. Then

$$\begin{aligned} E_1 &= \{|f(u_n) - f(u_1)|, |f(u_1) - f(u_2)|\} \\ &= \{|F_0 - F_{m+n}|, |F_{m+n} - F_{m+n-2}|\} = \{F_{m+n}, F_{m+n-1}\}. \end{aligned}$$

For the edge labeling between the vertex u_2 and starting vertex u_3 of the first loop, let $E_2 = \{f^*(u_2 u_3)\}$. Then

$$E_2 = \{|f(u_2) - f(u_3)|\} = \{|F_{m+n-2} - F_{m+n\_1}|\} = \{F_{m+n-3}\}.$$

For $l = 1$, let $E_3 = \{f^*(u_{i+2} u_{i+3}) : 1 \leq i \leq 2\}$. Then

$$\begin{aligned} E_3 &= \{|f(u_{i+2}) - f(u_{i+3})| : 1 \leq i \leq 2\} \\ &= \{|f(u_3) - f(u_4)|, |f(u_4) - f(u_5)|\} \\ &= \{|F_{m+n-1} - F_{m+n-3}|, |F_{m+n-3} - F_{m+n-5})|\} = \{F_{m+n-2}, F_{m+n-4}\}. \end{aligned}$$

For the edge labeling between the end vertex u_5 of the first loop and starting vertex u_6 of the second loop, let $E_3^{(1)} = \{f^*(u_5 u_6)\}$. Then

$$E_3^{(1)} = \{|f(u_5) - f(u_6)|\} = \{|F_{m+n-5} - F_{m+n-4}|\} = \{F_{m+n-6}\}.$$

For $l = 2$, let $E_4 = \{f^*(u_{i+2} u_{i+3}) : 4 \leq i \leq 5\}$. Then

$$\begin{aligned} E_4 &= \{|f(u_{i+2}) - f(u_{i+3})| : 4 \leq i \leq 5\} = \{|f(u_6) - f(u_7)|, |f(u_7) - f(u_8)|\} \\ &= \{|F_{m+n-4} - F_{m+n-6}|, |F_{m+n-6} - F_{m+n-8}|\} = \{F_{m+n-5}, F_{m+n-7}\}. \end{aligned}$$

For the edge labeling between the end vertex u_8 of the second loop and starting vertex u_9 of the third loop, let $E_4^{(1)} = \{f^*(u_8 u_9)\}$. Then

$$E_4^{(1)} = \{|f(u_8) - f(u_9)|\} = \{|F_{m+n-8} - F_{m+n-7}|\} = \quad \{F_{m+n-9}\},$$

etc.. For $l = \dfrac{n-3}{3} - 1$, let $E_{\frac{n-3}{3}-1} = \{f^*(u_{i+2} u_{i+3}) : n-8 \leq i \leq n-7\}$. Then

$$\begin{aligned} E_{\frac{n-3}{3}-1} &= \{|f(u_{i+2}) - f(u_{i+3})| : n-8 \leq i \leq n-7\} \\ &= \{|f(u_{n-6}) - f(u_{n-5})|, |f(u_{n-5}) - f(u_{n-4})|\} \\ &= \{|F_{m+8} - F_{m+6}|, |F_{m+6} - F_{m+4})|\} = \{F_{m+7}, F_{m+5}\}. \end{aligned}$$

For finding the edge labeling between the end vertex u_{n-4} of the $(\dfrac{n-3}{3} - 1)^{th}$ loop and starting vertex u_{n-3} of the $(\dfrac{n-3}{3})^{rd}$ loop, let $E^{(1)}_{\frac{n-3}{3}-1} = \{f^*(u_{n-4} u_{n-3})\}$. Then

$$E^{(1)}_{\frac{n-3}{3}-1} = \{|f(u_{n-4}) - f(u_{n-3})|\} = \{|F_{m+4} - F_{m+5}|\} = \{F_{m+3}\}.$$

For $l = \dfrac{n-3}{3}$, let $E_{\frac{n-3}{3}} = \{f^*(u_{i+2} u_{i+3}) : n-5 \leq i \leq n-4\}$. Then

$$\begin{aligned} E_{\frac{n-3}{3}} &= \{|f(u_{i+2}) - f(u_{i+3})| : n-5 \leq i \leq n-4\} \\ &= \{|f(u_{n-3}) - f(u_{n-2})|, |f(u_{n-2}) - f(u_{n-1})|\} \\ &= \{|F_{m+5} - F_{m+3}|, |F_{m+3} - F_{m+1}|\} = \{F_{m+4}, F_{m+2}\}. \end{aligned}$$

Let $E_1^* = \{f^*(u_{n-1}u_n)\}$ and $E_2^* = \{f^*(u_nv_1), f^*(v_1v_2)\}$. Then

$$E_1^* = \{|f(u_{n-1}) - f(u_n)|\} = \{|F_{m+1} - F_0|\} = \{F_{m+1}\},$$

$$\begin{aligned} E_2^* &= \{|f(u_n) - f(v_1)|, |f(v_1) - f(v_2)|\} \\ &= \{|F_0 - F_m|, |F_m - F_{m-2}|\} = \{F_m, F_{m-1}\}. \end{aligned}$$

For finding the edge labeling between the vertex v_2 and starting vertex v_3 of the first loop, let $E_3^* = \{f^*(v_2v_3)\}$. Then

$$E_3^* = \{|f(v_2) - f(v_3)|\} = \{|F_{m-2} - F_{m-1}|\} = \{F_{m-3}\}.$$

For $l = 1$, let $E_4^* = \{f^*(v_{i+2}v_{i+3}) : 1 \le i \le 2\}$. Then

$$\begin{aligned} E_4^* &= \{|f(v_{i+2}) - f(v_{i+3})| : 1 \le i \le 2\} \\ &= \{|f(v_3) - f(v_4)|, |f(v_4) - f(v_5)|\} \\ &= \{|F_{m-1} - F_{m-3}|, |F_{m-3} - F_{m-5}|\} = \{F_{m-2}, F_{m-4}\}. \end{aligned}$$

Now let $E_4^{(*1)} = \{f^*(v_5v_6)\}$. Then

$$E_4^{(*1)} = \{|f(v_5) - f(v_6)|\} = \{|F_{m-5} - F_{m-4}|\} = \{F_{m-6}\}.$$

For $l = 2$, let $E_5^* = \{f^*(v_{i+2}v_{i+3}) : 4 \le i \le 5\}$. Calculation shows that

$$\begin{aligned} E_5^* &= \{|f(v_{i+2}) - f(v_{i+3})| : 4 \le i \le 5\} \\ &= \{|f(v_6) - f(v_7)|, |f(v_7) - f(v_8)|\} \\ &= \{|F_{m-4} - F_{m-6}|, |F_{m-6} - F_{m-8}|\} = \{F_{m-5}, F_{m-7}\}. \end{aligned}$$

Let $E_5^{(*1)} = \{f^*(v_8v_9)\}$. We find the edge labeling between the end vertex v_8 of the second loop and starting vertex v_9 of the third loop. In fact,

$$E_5^{(*1)} = \{|f(v_8) - f(v_9)|\} = \{|F_{m-8} - F_{m-7}|\} = \{F_{m-9}\}$$

etc.. For $l = \dfrac{m-5}{3} - 1$, let $E^*_{\frac{m-5}{3}-1} = \{f^*(v_{i+2}v_{i+3}) : m - 10 \le i \le m - 9\}$. Then

$$\begin{aligned} E^*_{\frac{m-5}{3}-1} &= \{|f(v_{i+2}) - f(v_{i+3})| : m - 10 \le i \le m - 9\} \\ &= \{|f(v_{m-8}) - f(v_{m-7})|, |f(v_{m-7}) - f(v_{m-6})|\} \\ &= \{|F_{10} - F_8|, |F_8 - F_6|\} = \{F_9, F_7\}. \end{aligned}$$

Similarly, for finding the edge labeling between the end vertex v_{m-6} of the $(\dfrac{m-5}{3} - 1)^{th}$ loop and starting vertex v_{m-5} of the $(\dfrac{m-5}{3})^{rd}$ loop, let $E^{(*1)}_{\frac{m-5}{3}-1} = \{f^*(v_{m-6}v_{m-5})\}$. Then

$$E^{(*1)}_{\frac{m-5}{3}-1} = \{|f(v_{m-6}) - f(v_{m-5})|\} = \{|F_6 - F_7|\} = \{F_5\}.$$

For $l = \dfrac{m-5}{3}$, let $E^*_{\frac{m-5}{3}} = \{f^*(v_{i+2}v_{i+3}) : m-7 \leq i \leq m-6\}$. Then

$$\begin{aligned}
E^*_{\frac{m-5}{3}} &= \{|f(v_{i+2}) - f(v_{i+3})| : m-7 \leq i \leq m-6\} \\
&= \{|f(v_{m-5}) - f(v_{m-4})|, |f(v_{m-4}) - f(v_{m-3})|\} \\
&= \{|F_7 - F_5|, |F_5 - F_3|\} = \{F_6, F_4\}.
\end{aligned}$$

For $l = \dfrac{m-4}{3} - 1$, let $E^*_{\frac{m-4}{3}-1} = \{f^*(v_{i+2}v_{i+3}) : m-9 \leq i \leq m-8\}$. We find that

$$\begin{aligned}
E^*_{\frac{m-4}{3}-1} &= \{|f(v_{i+2}) - f(v_{i+3})| : m-9 \leq i \leq m-8\} \\
&= \{|f(v_{m-7}) - f(v_{m-6})|, |f(v_{m-6}) - f(v_{m-5})|\} \\
&= \{|F_9 - F_7|, |F_7 - F_5|\} = \{F_8, F_6\}.
\end{aligned}$$

For getting the edge labeling between the end vertex v_{m-5} of the $(\dfrac{m-4}{3} - 1)^{th}$ loop and starting vertex v_{m-4} of the $(\dfrac{m-4}{3})^{rd}$ loop, let $E^{(*1)}_{\frac{m-4}{3}-1} = \{f^*(v_{m-5}v_{m-4})\}$. Then

$$E^{(*1)}_{\frac{m-4}{3}-1} = \{|f(v_{m-5}) - f(v_{m-4})|\} = \{|F_5 - F_6|\} = \{F_4\}.$$

For $l = \dfrac{m-4}{3}$, let $E^*_{\frac{m-4}{3}} = \{f^*(v_{i+2}v_{i+3}) : m-6 \leq i \leq m-5\}$. Then

$$\begin{aligned}
E^*_{\frac{m-4}{3}} &= \{|f(v_{i+2}) - f(v_{i+3})| : m-6 \leq i \leq m-5\} \\
&= \{|f(v_{m-4}) - f(v_{m-3})|, |f(v_{m-3}) - f(v_{m-2})|\} \\
&= \{|F_6 - F_4|, |F_4 - F_2|\} = \{F_5, F_3\}.
\end{aligned}$$

For $l = \dfrac{m-3}{3} - 1$, let $E^*_{\frac{m-3}{3}-1} = \{f^*(v_{i+2}v_{i+3}) : m-8 \leq i \leq m-7\}$. Then

$$\begin{aligned}
E^*_{\frac{m-3}{3}-1} &= \{|f(v_{i+2}) - f(v_{i+3})| : m-8 \leq i \leq m-7\} \\
&= \{|f(v_{m-5}) - f(v_{m-4})|, |f(v_{m-4}) - f(v_{m-3})|\} \\
&= \{|F_8 - F_6|, |F_6 - F_4|\} = \{F_7, F_5\}.
\end{aligned}$$

For the edge labeling between the end vertex v_{m-3} of the $(\dfrac{m-3}{3} - 1)^{th}$ loop and starting vertex v_{m-2} of the $(\dfrac{m-3}{3})^{rd}$ loop, let $E^{(*1)}_{\frac{m-3}{3}-1} = \{f^*(v_{m-3}v_{m-2})\}$. Then

$$E^{(*1)}_{\frac{m-3}{3}-1} = \{|f(v_{m-3}) - f(v_{m-2})|\} = \{|F_4 - F_5|\} = \{F_3\}.$$

Similarly, for $l = \dfrac{m-3}{3}$, let $E^*_{\frac{m-3}{3}} = \{f^*(v_{i+2}v_{i+3}) : m-5 \leq i \leq m-4\}$. Then

$$\begin{aligned}
E^*_{\frac{m-3}{3}} &= \{|f(v_{i+2}) - f(v_{i+3})| : m-5 \leq i \leq m-4\} \\
&= \{|f(v_{m-3}) - f(v_{m-2})|, |f(v_{m-2}) - f(v_{m-1})|\} \\
&= \{|F_5 - F_3|, |F_3 - F_1|\} = \{F_4, F_2\}.
\end{aligned}$$

Now let

$$\begin{aligned}
E^{(1)} &= \left(E_1 \bigcup E_2 \bigcup \cdots \bigcup E_{\frac{n-3}{3}}\right) \bigcup \left(E^1_3 \bigcup E^1_4 \bigcup \cdots \bigcup E^1_{\frac{n-3}{3}}\right) \\
&\quad \bigcup \left(E^*_1 \bigcup E^*_2 \bigcup \cdots \bigcup E^*_{\frac{m-3}{3}}\right) \bigcup \left(E^{(*1)}_4 \bigcup E^{(*1)}_5 \bigcup \cdots \bigcup E^{(*1)}_{\frac{m-3}{3}-1}\right),
\end{aligned}$$

$$\begin{aligned} E^{(2)} &= \left(E_1 \bigcup E_2 \bigcup \cdots \bigcup E_{\frac{n-3}{3}}\right) \bigcup \left(E_3^1 \bigcup E_4^1 \bigcup \cdots \bigcup E^1_{\frac{n-3}{3}}\right) \\ &\bigcup \left(E_1^* \bigcup E_2^* \bigcup \cdots \bigcup E^*_{\frac{m-4}{3}}\right) \bigcup \left(E_4^{(*1)} \bigcup E_5^{(*1)} \bigcup \cdots \bigcup E^{(*1)}_{\frac{m-4}{3}-1}\right) \end{aligned}$$

and

$$\begin{aligned} E^{(3)} &= \left(E_1 \bigcup E_2 \bigcup \cdots \bigcup E_{\frac{n-3}{3}}\right) \bigcup \left(E_3^1 \bigcup E_4^1 \bigcup \cdots \bigcup E^1_{\frac{n-3}{3}}\right) \\ &\bigcup \left(E_1^* \bigcup E_2^* \bigcup \cdots \bigcup E^*_{\frac{m-5}{3}}\right) \bigcup \left(E_4^{(*1)} \bigcup E_5^{(*1)} \bigcup \cdots \bigcup E^{(*1)}_{\frac{m-5}{3}-1}\right). \end{aligned}$$

If $m \equiv 0(mod3)$, let $E_1^{**} = \{f^*(v_{m-1}v_m)\}$, then

$$E_1^{**} = \{|f(v_{m-1} - f(v_m)|\} = \{|F_1 - F_2|\} = \{F_1\}.$$

Thus,

$$E = E_1^{**} \cup E^{(1)} = \{F_1, F_2, \ldots, F_{m+n}\}.$$

For example the super Fibonacci graceful labeling of $C_6 \oplus P_6$ is shown in Fig.4.

$C_6 \oplus P_6$:

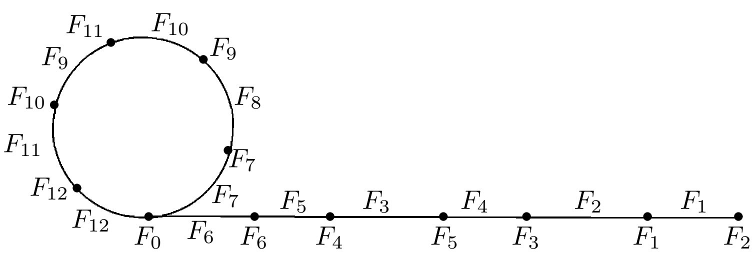

Fig.4

If $m \equiv 1(mod3)$, let $E_2^{**} = \{f^*(v_{m-2}v_{m-1}), f^*(v_{m-1}v_m)\}$, then

$$\begin{aligned} E_2^{**} &= \{|f(v_{m-2} - f(v_{m-1}|, |f(v_{m-1} - f(v_m)|\} \\ &= \{|F_2 - F_3|\,|F_3 - F_1|\} = \{F_1, F_2\}. \end{aligned}$$

Thus,

$$E = E_2^{**} \cup E^{(2)} = \{F_1, F_2, \ldots, F_{m+n}\}.$$

For example the super Fibonacci graceful labeling of $C_6 \oplus P_7$ is shown in Fig.5.

$C_6 \oplus P_7$:

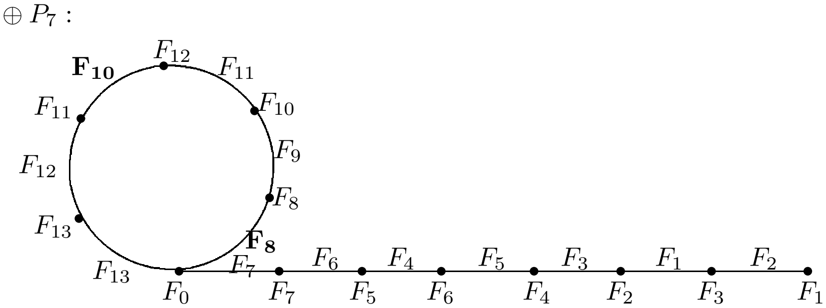

Fig.5

If $m \equiv 2(mod 3)$, let $E_3^{**} = \{f^*(v_{m-3}v_{m-2}), f^*(v_{m-2}v_{m-1}), f^*(v_{m-1}v_m)\}$, then

$$\begin{aligned} E_3^{**} &= \{|f(v_{m-3}) - f(v_{m-2})|, |f(v_{m-2}) - f(v_{m-1})|, |f(v_{m-1} - f(v_m)|\} \\ &= \{|F_3 - F_4|, |F_4 - F_2|, |F_2 - F_1|\} = \{F_2, F_3, F_1\}. \end{aligned}$$

Thus,

$$E = E_3^{**} \cup E^{(3)} = \{F_1, F_2, \ldots, F_{m+n}\}.$$

For example the super Fibonacci graceful labeling of $C_6 \oplus P_5$ is shown in Fig.6.

$C_6 \oplus P_5$:

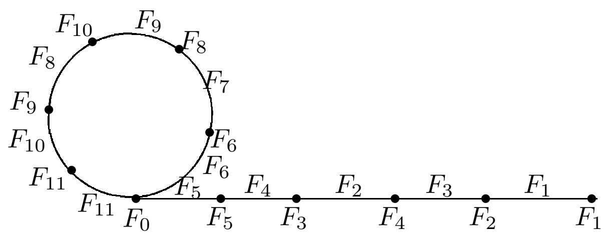

Fig.6

Therefore, $C_n \oplus P_m$ admits a super Fibonacci graceful labeling. Hence, $C_n \oplus P_m$ is a super Fibonacci graceful graph. □

Definition 2.6 *The graph $G = F_n \oplus K_{1,m}^+$ consists of a fan F_n and the extension graph of $K_{1,m}^+$ which is attached with the maximum degree of the vertex of F_n.*

Theorem 2.7 *The graph $G = F_n \oplus K_{1,m}^+$ is a super Fibonacci graceful graph.*

Proof Let $V(G) = U \cup V$, where $U = \{u_0, u_1, \ldots, u_n\}$ be the vertex set of F_n and $V = (V_1, V_2)$ be the bipartion of $K_{1,m}$, where $V_1 = \{v = u_0\}$ and $V_2 = \{v_1, v_2, \ldots, v_m\}$ and $w_1, w_2, \ldots, w_m$ be the pendant vertices joined with $v_1, v_2, \ldots, v_m$ respectively. Also, $|V(G)| = 2m + n + 1$ and $|E(G)| = 2m + 2n - 1$.

Case 1 m, n is even.

Define $f : V(G) \to \{F_0, F_1, \ldots, F_q\}$ by $f(u_0) = F_0$, $f(u_i) = F_{2m+2n-1-2(i-1)}$ if $1 \le i \le n$; $f(v_{2i-1}) = F_{2m-4(i-1)}$ if $1 \le i \le \dfrac{m}{2}$; $f(v_{2i}) = F_{2m-3-4(i-1)}$ if $1 \le i \le \dfrac{m}{2}$; $f(w_{2i-1}) = F_{2m-2-4(i-1)}$ if $1 \le i \le \dfrac{m}{2}$ and $f(w_{2i}) = F_{2m-1-4(i-1)}$ if $1 \le i \le \dfrac{m}{2}$.

We claim that all these edge labels are distinct. Calculation shows that

$$\begin{aligned} E_1 &= \{f^*(u_iu_{i+1}) : i = 1, 2, \ldots, n-1\} \\ &= \{|f(u_i) - f(u_{i+1})| : i = 1, 2, \ldots, n-1\} \\ &= \{|f(u_1) - f(u_2)|, |f(u_2) - f(u_3)|, \ldots, |f(u_{n-1}) - f(u_n)|\} \\ &= \{|F_{2n+2m-1} - F_{2n+2m-3}|, |F_{2n+2m-3} - F_{2n+2m-5}|, \ldots, |F_{2m+3} - F_{2m+1}|\} \\ &= \{F_{2n+2m-2}, F_{2n+2m-4}, \ldots, F_{2m+4}, F_{2m+2}\}, \end{aligned}$$

$$\begin{aligned}
E_2 &= \{f^*(u_0u_i) : i = 1, 2, \dots, n\} \\
&= \{|f(u_0) - f(u_i)| : i = 1, 2, \dots, n\} \\
&= \{|f(u_0) - f(u_1)|, |f(u_0) - f(u_2)|, \dots, |f(u_0) - f(u_{n-1})|, |f(u_0) - f(u_n)|\} \\
&= \{|F_0 - F_{2n+2m-1}|, |F_0 - F_{2n+2m-3}|, \dots, |F_0 - F_{2m+3}|, |F_0 - F_{2m+1}|\} \\
&= \{F_{2n+2m-1}, F_{2n+2m-3}, \dots, F_{2m+3}, F_{2m+1}\},
\end{aligned}$$

$$\begin{aligned}
E_3 &= \{f^*(u_0v_{2i-1}) : 1 \le i \le \frac{m}{2}\} \\
&= \{|f(u_0) - f(v_{2i-1})| : 1 \le i \le \frac{m}{2}\} \\
&= \{|f(u_0) - f(v_1)|, |f(u_0) - f(v_3)|, \dots, |f(u_o) - f(v_{m-3})|, |f(u_0) - f(v_{m-1})|\} \\
&= \{|F_0 - F_{2m}|, |F_0 - F_{2m-4}|, \dots, |F_0 - F_8|, |F_0 - F_4|\} \\
&= \{F_{2m}, F_{2m-4}, \dots, F_8, F_4\},
\end{aligned}$$

$$\begin{aligned}
E_4 &= \{f^*(u_0v_{2i}) : 1 \le i \le \frac{m}{2}\} \\
&= \{|f(u_0) - f(v_{2i})| : 1 \le i \le \frac{m}{2}\} \\
&= \{|f(u_0) - f(v_2)|, |f(u_0) - f(v_4)|, \dots, |f(u_0) - f(v_{m-2})|, |f(u_0) - f(v_m)|\} \\
&= \{|F_0 - F_{2m-3}|, |F_0 - F_{2m-7}|, \dots, |F_0 - F_5|, |F_0 - F_1|\} \\
&= \{F_{2m-3}, F_{2m-7}, \dots, F_5, F_1\},
\end{aligned}$$

$$\begin{aligned}
E_5 &= \{f^*(v_{2i-1}w_{2i-1}) : 1 \le i \le \frac{m}{2}\} \\
&= \{|f(v_{2i-1}) - f(w_{2i-1})| : 1 \le i \le \frac{m}{2}\} \\
&= \{|f(v_1) - f(w_1)|, |f(v_3) - f(w_3)|, \dots, |f(v_{m-3}) - f(w_{m-3})|, |f(v_{m-1}) - f(w_{m-1})|\} \\
&= \{|F_{2m} - F_{2m-2}|, |F_{2m-4} - F_{2m-6}|, \dots, |F_8 - F_6|, |F_4 - F_2|\} \\
&= \{F_{2m-1}, F_{2m-5} \dots, F_7, F_3\},
\end{aligned}$$

$$\begin{aligned}
E_6 &= \{f^*(v_{2i}w_{2i}) : 1 \le i \le \frac{m}{2}\} \\
&= \{|f(v_{2i}) - f(w_{2i})| : 1 \le i \le \frac{m}{2}\} \\
&= \{|f(v_2) - f(w_2)|, |f(v_4) - f(w_4)|, \dots, |f(v_{m-2}) - f(w_{m-2})|, |f(v_m) - f(w_m)| \\
&= \{|F_{2m-3} - F_{2m-1}|, |F_{2m-7} - F_{2m-5}|, \dots, |F_5 - F_7|, |F_1 - F_3|\} \\
&= \{F_{2m-2}, F_{2m-6} \dots, F_6, F_2\}.
\end{aligned}$$

Therefore,

$$E = E_1 \bigcup E_2 \bigcup \cdots \bigcup E_6 = \{F_1, F_2, \dots, F_{2m+2n-1}\}.$$

Thus, the edge labels are distinct. Therefore, $F_n \oplus K_{1,m}^+$ admits super Fibonacci graceful labeling. Hence, $F_n \oplus K_{1,m}^+$ is a super Fibonacci graceful graph.

For example the super Fibonacci graceful labeling of $F_4 \oplus K_{1,4}^+$ is shown in Fig.7.

$F_4 \oplus K_{1,4}^{+}$:

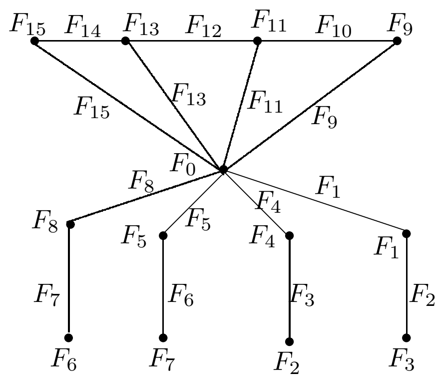

Fig.7

Case 2 m even, n odd.

Proof of this case is analogous to case(i).

For example the super Fibonacci graceful labeling of $F_5 \oplus K_{1,4}^{+}$ is shown in Fig.8.

$F_5 \oplus K_{1,4}^{+}$:

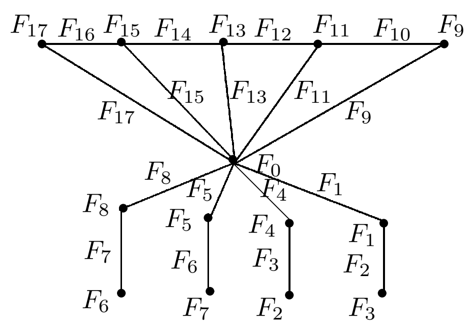

Fig.8

Case 3 m, n is odd.

Define $f : V(G) \to \{F_0, F_1, \ldots, F_q\}$ by $f(u_0) = F_0$; $f(u_i) = F_{2m+2n-1-2(i-1)}$ if $1 \leq i \leq n$; $f(w_m) = F_1$; $f(v_{2i-1}) = F_{2m-4(i-1)}$ if $1 \leq i \leq \dfrac{m+1}{2}$; $f(v_{2i}) = F_{2m-3-4(i-1)}$ if $1 \leq i \leq \dfrac{m-1}{2}$; $f(w_{2i-1}) = F_{2m-2-4(i-1)}$ if $1 \leq i \leq \dfrac{m-1}{2}$ and $f(w_{2i}) = F_{2m-1-4(i-1)}$ if $1 \leq i \leq \dfrac{m-1}{2}$.

We claim that the edge labels are distinct. Calculation shows that

$$\begin{aligned}
E_1 &= \{f^*(u_i u_{i+1}) : i = 1, 2, \ldots, n-1\} \\
&= \{|f(u_i) - f(u_{i+1})| : i = 1, 2, \ldots, n-1\} \\
&= \{|f(u_1) - f(u_2)|, |f(u_2) - f(u_3)|, \ldots, |f(u_{n-2}) - f(u_{n-1})|, |f(u_{n-1}) - f(u_n)|\} \\
&= \{|F_{2n+2m-1} - F_{2n+2m-3}|, |F_{2n+2m-3} - F_{2n+2m-5}|, \ldots, |F_{2m+5} - F_{2m+3}|, \\
&\quad |F_{2m+3} - F_{2m+1}|\} = \{F_{2n+2m-2}, F_{2n+2m-4}, \ldots, F_{2m+4}, F_{2m+2}\},
\end{aligned}$$

$$\begin{aligned}
E_2 &= \{f^*(u_0u_i) : i = 1, 2, \ldots, n\} \\
&= \{|f(u_0) - f(u_i)| : i = 1, 2, \ldots, n\} \\
&= \{|f(u_0) - f(u_1)|, |f(u_0) - f(u_2)|, \ldots, |f(u_0) - f(u_{n-1})|, |f(u_0) - f(u_n)|\} \\
&= \{|F_0 - F_{2n+2m-1}|, |F_0 - F_{2n+2m-3}|, \ldots, |F_0 - F_{2m+3}|, |F_0 - F_{2m+1}|\} \\
&= \{F_{2n+2m-1}, F_{2n+2m-3}, \ldots, F_{2m+3}, F_{2m+1}\},
\end{aligned}$$

$$\begin{aligned}
E_3 &= \{f^*(u_0v_{2i-1}) : 1 \leq i \leq \frac{m+1}{2}\} \\
&= \{|f(u_0) - f(v_{2i-1})| : 1 \leq i \leq \frac{m+1}{2}\} \\
&= \{|f(u_0) - f(v_1)|, |f(u_0) - f(v_3)|, \ldots, |f(u_o) - f(v_{m-2})|, |f(u_0) - f(v_m)|\} \\
&= \{|F_0 - F_{2m}|, |F_0 - F_{2m-4}|, \ldots, |F_0 - F_6|, |F_0 - F_2|\} \\
&= \{F_{2m}, F_{2m-4}, \ldots, F_6, F_2\},
\end{aligned}$$

$$\begin{aligned}
E_4 &= \{f^*(u_0v_{2i}) : 1 \leq i \leq \frac{m-1}{2}\} \\
&= \{|f(u_0) - f(v_{2i})| : 1 \leq i \leq \frac{m-1}{2}\} \\
&= \{|f(u_0) - f(v_2)|, |f(u_0) - f(v_4)|, \ldots, |f(u_o) - f(v_{m-3})|, |f(u_0) - f(v_{m-1})|\} \\
&= \{|F_0 - F_{2m-3}|, |F_0 - F_{2m-7}|, \ldots, |F_0 - F_7|, |F_0 - F_3|\} \\
&= \{F_{2m-3}, F_{2m-7}. \ldots, F_7, F_3\},
\end{aligned}$$

$$E_5 = \{f^*(v_mw_m)\} = \{|f(v_m) - f(w_m)|\} = \{|F_2 - F_1|\} = \{F_1\},$$

$$\begin{aligned}
E_6 &= \{f^*(v_{2i-1}w_{2i-1}) : 1 \leq i \leq \frac{m-1}{2}\} \\
&= \{|f(v_{2i-1}) - f(w_{2i-1})| : 1 \leq i \leq \frac{m-1}{2}\} \\
&= \{|f(v_1) - f(w_1)|, |f(v_3) - f(w_3)|, \ldots, |f(v_{m-4}) - f(w_{m-4})|, |f(v_{m-2}) - f(w_{m-2})|\} \\
&= \{|F_{2m} - F_{2m-2}|, |F_{2m-4} - F_{2m-6}|, \ldots, |F_6 - F_8|, |F_6 - F_4|\} \\
&= \{F_{2m-1}, F_{2m-5}. \ldots, F_9, F_5\},
\end{aligned}$$

$$\begin{aligned}
E_7 &= \{f^*(v_{2i}w_{2i}) : 1 \leq i \leq \frac{m-1}{2}\} \\
&= \{|f(v_{2i}) - f(w_{2i})| : 1 \leq i \leq \frac{m-1}{2}\} \\
&= \{|f(v_2) - f(w_2)|, |f(v_4) - f(w_4)|, \ldots, |f(v_{m-3}) - f(w_{m-3})|, |f(v_{m-1}) - f(w_{m-1})|\} \\
&= \{|F_{2m-3} - F_{2m-1}|, |F_{2m-7} - F_{2m-5}|, \ldots, |F_7 - F_9|, |F_3 - F_5|\} \\
&= \{F_{2m-2}, F_{2m-6}. \ldots, F_8, F_4\}.
\end{aligned}$$

Therefore,

$$E = E_1 \bigcup E_2 \bigcup \cdots \bigcup E_7 = \{F_1, F_2, \ldots, F_{2m+2n-1}\}.$$

Thus, the edge labels are distinct.Therefore, $F_n \oplus K^+_{1,m}$ admits super Fibonacci graceful labeling. Whence, $F_n \oplus K^+_{1,m}$ is a super Fibonacci graceful graph.

For example the super Fibonacci graceful labeling of $F_5 \oplus K^+_{1,3}$ is shown in Fig.9.

$F_5 \oplus K^+_{1,3}$:

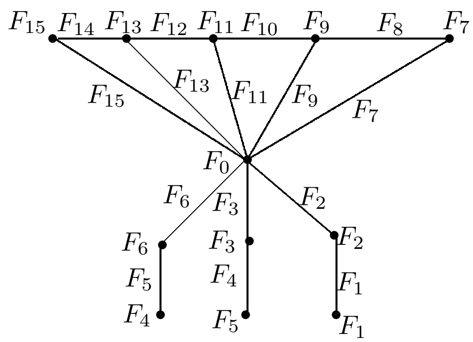

Fig.9

Case 4 m odd, n even.

Proof of this case is analogous to Case 4.

For example the super Fibonacci graceful labeling of $F_4 \oplus K^+_{1,3}$ is shown in Fig.10.

$F_4 \oplus K^+_{1,3}$:

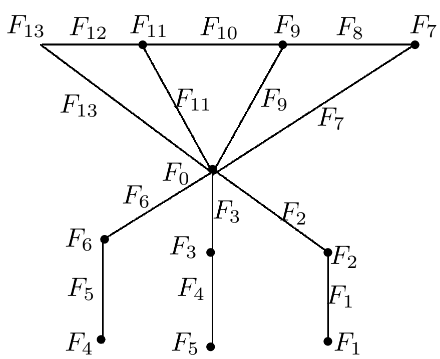

Fig.10

□

Definition 2.8 *The graph $G = C_n \oplus K_{1,m}$ consists of a cycle C_n of length n and a star $K_{1,m}$ is attached with the vertex u_n of C_n.*

Theorem 2.9 *The graph $G = C_n \oplus K_{1,m}$ is a super Fibonacci graceful graph when $n \equiv 0(mod3)$.*

Proof Let $V(G) = V_1 \cup V_2$, where $V_1 = \{u_1, u_2, \ldots, u_n\}$ be the vertex set of C_n and $V_2 = \{v = u_n, v_1, v_2, \ldots, v_m\}$ be the vertex set of $K_{1,m}$. Also, $|V(G)| = |E(G)| = m+n$. Define $f : V(G) \to \{F_0, F_1, F_2, \ldots, F_q\}$ by $f(u_n) = F_0$; $f(u_i) = F_{m+n-2(i-1)}$ if $1 \le i \le 2$; $f(v_i) = F_i$ if $1 \le i \le m$ and for $l = 1, 2, \ldots, \dfrac{n-3}{3}$, $f(u_{i+2}) = F_{m+n-1-2(i-1)+3(l-1)}$ if $3l - 2 \le i \le 3l$.

We claim that the edge labels are distinct. Calculation shows that

$$\begin{aligned} E_1 &= \{f^*(u_nv_i) : 1 \leq i \leq m\} \\ &= \{|f(u_n) - f(v_i)| : 1 \leq i \leq m\} \\ &= \{|f(u_n) - f(v_1)|, |f(u_n) - f(v_2)|, \ldots, |f(u_n) - f(v_{m-1})|, |f(u_n) - f(v_m)|\} \\ &= \{|F_0 - F_1|, |F_0 - F_2|, \ldots, |F_0 - F_{m-1}|, |F_0 - F_m|\} \\ &= \{F_1, F_2, \ldots, F_{m-1}, F_m\}, \end{aligned}$$

$$\begin{aligned} E_2 &= \{f^*(u_nu_1), f^*(u_1u_2)\} = \{|f(u_n) - f(u_1)|, |f(u_1) - f(u_2)|\} \\ &= \{|F_0 - F_{m+n}|, |F_{m+n} - F_{m+n-2}|\} = \{F_{m+n}, F_{m+n-1}\}. \end{aligned}$$

For the edge labeling between the vertex u_2 and starting vertex u_3 of the first loop, let $E_3 = \{f^*(u_2u_3)\}$. Then

$$E_3 = \{|f(u_2) - f(u_3)|\} = \{|F_{m+n-2} - F_{m+n-1}\} = \{F_{m+n-3}\}.$$

For $l = 1$, let $E_4 = \{f^*(u_{i+2}u_{i+3}) : 1 \leq i \leq 2\}$. Then

$$\begin{aligned} E_4 &= \{|f(u_{i+2}) - f(u_{i+3})| : 1 \leq i \leq 2\} \\ &= \{|f(u_3) - f(u_4)|, |f(u_4) - f(u_5)|\} \\ &= \{|F_{m+n-1} - F_{m+n-3}|, |F_{m+n-3} - F_{m+n-5})|\} \\ &= \{F_{m+n-2}, F_{m+n-4}\}. \end{aligned}$$

Let $E_4^{(1)} = \{f^*(u_5u_6)\}$. Then

$$E_4^{(1)} = \{|f(u_5) - f(u_6)|\} = \{|F_{m+n-5} - F_{m+n-4}|\} = \{F_{m+n-6}\}.$$

For $l = 2$, let $E_5 = \{f^*(u_{i+2}u_{i+3}) : 4 \leq i \leq 5\}$. Then

$$\begin{aligned} E_5 &= \{|f(u_{i+2}) - f(u_{i+3})| : 4 \leq i \leq 5\} \\ &= \{|f(u_6) - f(u_7)|, |f(u_7) - f(u_8)|\} \\ &= \{|F_{m+n-4} - F_{m+n-6}|, |F_{m+n-6} - F_{m+n-8})|\} \\ &= \{F_{m+n-5}, F_{m+n-7}\}. \end{aligned}$$

For finding the edge labeling between the end vertex u_8 of the second loop and starting vertex u_9 of the third loop, let $E_5^{(1)} = \{f^*(u_8u_9)\}$. Then

$$E_5^{(1)} = \{|f(u_8) - f(u_9)|\} = \{|F_{m+n-8} - F_{m+n-7}|\} = \{F_{m+n-9}\}$$

etc.. Similarly, for $l = \dfrac{n-3}{3} - 1$, let $E_{\frac{n-3}{3}-1} = \{f^*(u_{i+2}u_{i+3}) : n-8 \leq i \leq n-7\}$. Then

$$\begin{aligned} E_{\frac{n-3}{3}-1} &= \{|f(u_{i+2}) - f(u_{i+3})| : n-8 \leq i \leq n-7\} \\ &= \{|f(u_{n-6}) - f(u_{n-5})|, |f(u_{n-5}) - f(u_{n-4})|\} \\ &= \{|F_{m+8} - F_{m+6}|, |F_{m+6} - F_{m+4})|\} = \{F_{m+7}, F_{m+5}\}. \end{aligned}$$

For finding the edge labeling between the end vertex u_{n-4} of the $(\dfrac{n-3}{3}-1)^{th}$ loop and starting vertex u_{n-3} of the $(\dfrac{n-3}{3})^{rd}$ loop, let $E^{(1)}_{\frac{n-3}{3}-1}=\{f^*(u_{n-4}u_{n-3})\}$. Then

$$E^{(1)}_{\frac{n-3}{3}-1}=\{|f(u_{n-4})-f(u_{n-3})|\}=\{|F_{m+4}-F_{m+5}|\}=\{F_{m+3}\}.$$

For $l=\dfrac{n-3}{3}$, let $E_{\frac{n-3}{3}}=\{f^*(u_{i+2}u_{i+3}):n-5\le i\le n-4\}$. Then

$$\begin{aligned}E_{\frac{n-3}{3}} &= \{|f(u_{i+2})-f(u_{i+3})|:n-5\le i\le n-4\}\\ &= \{|f(u_{n-3})-f(u_{n-2})|,|f(u_{n-2})-f(u_{n-1})|\}\\ &= \{|F_{m+5}-F_{m+3}|,|F_{m+3}-F_{m+1})|\}=\{F_{m+4},F_{m+2}\}.\end{aligned}$$

Let $E^*_1=\{f^*(u_{n-1}u_n)\}$. Then

$$E^*_1=\{|f(u_{n-1})-f(u_n)|\}=\{|F_{m+1}-F_0|\}=\{F_{m+1}\}.$$

Therefore,

$$\begin{aligned}E &= \left(E_1\bigcup E_2\bigcup\cdots\bigcup E_{\frac{n-3}{3}}\right)\bigcup\left(E^{(1)}_4\bigcup E^{(1)}_5\bigcup\cdots\bigcup E^{(1)}_{\frac{n-3}{3}-1}\right)\bigcup E^*_1\\ &= \{F_1,F_2,\ldots,F_{m+n}\}.\end{aligned}$$

Thus, all edge labels are distinct. Therefore, the graph $G=C_n\oplus K_{1,m}$ admits super Fibonacci graceful labeling. Whence, it is a super Fibonacci graceful graph. □

Example 2.10 This example shows that the graph $C_6\oplus K_{1,4}$ is a super Fibonacci graceful graph.

$C_6\oplus K_{1,4}$:

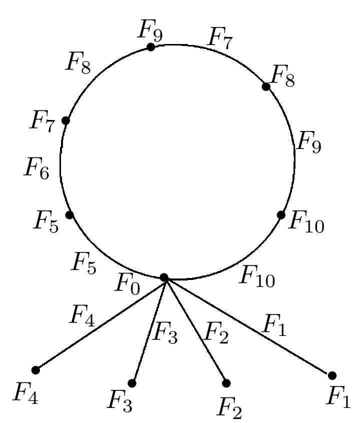

Fig.11

Definition 2.11 *$G=K_{1,n}\oslash K_{1,2}$ is a graph in which $K_{1,2}$ is joined with each pendant vertex of $K_{1,n}$.*

Theorem 2.12 *The graph $G=K_{1,n}\oslash K_{1,2}$ is a super Fibonacci graceful graph.*

Proof Let $\{u_0, u_1, u_2, \ldots, u_n\}$ be the vertex set of $K_{1,n}$ and $v_1, v_2, \ldots, v_n$ and $w_1, w_2, \ldots, w_n$ be the vertices joined with the pendant vertices $u_1, u_2, \ldots, u_n$ of $K_{1,n}$ respectively. Also, $|V(G)| = 3n+1$ and $|E(G)| = 3n$. Define $f : V(G) \to \{F_0, F_1, F_2, \ldots, F_q\}$ by $f(u_0) = F_0$, $f(u_i) = F_{3n-3(i-1)}$, $1 \le i \le n$, $f(v_i) = F_{3n-1-3(i-1)}$, $1 \le i \le n$, $f(w_i) = F_{3n-2-3(i-1)}$, $1 \le i \le n$.

We claim that the edge labels are distinct. Calculation shows that

$$\begin{aligned} E_1 &= \{f^*(u_0u_i) : i = 1, 2, \ldots, n\} \\ &= \{|f(u_0) - f(u_i)| : i = 1, 2, \ldots n\} \\ &= \{|f(u_0) - f(u_1)|, |f(u_0) - f(u_2)|, \ldots, |f(u_0) - f(u_{n-1})|, |f(u_0) - f(u_n)|\} \\ &= \{|F_0 - F_{3n}|, |F_0 - F_{3n-3}|, \ldots, |F_0 - F_6|, |F_0 - F_3|\} \\ &= \{F_{3n}, F_{3n-3}, \ldots, F_6, F_3\}, \end{aligned}$$

$$\begin{aligned} E_2 &= \{f^*(u_iv_i) : i = 1, 2, \ldots, n\} \\ &= \{|f(u_i) - f(v_i)| : i = 1, 2, \ldots, n\} \\ &= \{|f(u_1) - f(v_1)|, |f(u_2) - f(v_2)|, \ldots, |f(u_{n-1}) - f(v_{n-1})|, = |f(u_n) - f(v_n)|\} \\ &= \{|F_{3n} - F_{3n-1}|, |F_{3n-3} - F_{3n-4}|, \ldots, |F_6 - F_5|, |F_3 - F_2|\} \\ &= \{F_{3n-2}, F_{3n-5}, \ldots, F_4, F_1\}, \end{aligned}$$

$$\begin{aligned} E_3 &= \{f^*(u_iw_i) : i = 1, 2, \ldots, n\} \\ &= \{|f(u_i) - f(w_i)| : i = 1, 2, \ldots, n\} \\ &= 1\{|f(u_1) - f(w_1)|, |f(u_2) - f(w_2)|, \ldots, |f(u_{n-1}) - f(w_{n-1})|, |f(u_n) - f(w_n)|\} \\ &= \{|F_{3n} - F_{3n-2}|, |F_{3n-3} - F_{3n-5}|, \ldots, |F_6 - F_4|, |F_3 - F_1|\} \\ &= \{F_{3n-1}, F_{3n-4}, \ldots, F_5, F_2\}. \end{aligned}$$

Therefore,

$$E = E_1 \bigcup E_2 \bigcup E_3 = \{F_1, F_2, \ldots, F_{3n}\}.$$

Thus, all edge labels are distinct. Therefore, $K_{1,n} \oslash K_{1,2}$ admits super Fibonacci graceful labeling. Whence, it is a super Fibonacci graceful graph. □

Example 2.13 This example shows that the graph $K_{1,3} \oslash K_{1,2}$ is a super Fibonacci graceful graph.

$K_{1,3} \oslash K_{1,2}$:

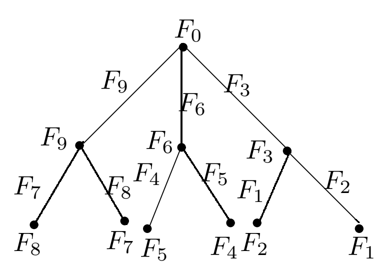

Fig.12

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A Note on
Smarandachely Consistent Symmetric n-Marked Graphs

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Abstract: A *Smarandachely k-marked graph* is an ordered pair $S = (G, \mu)$ where $G = (V, E)$ is a graph called *underlying graph of* S and $\mu : V \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k)$ is a function, where each $\overline{e}_i \in \{+, -\}$. An *$n$-tuple* $(a_1, a_2, ..., a_n)$ is *symmetric*, if $a_k = a_{n-k+1}, 1 \leq k \leq n$. Let $H_n = \{(a_1, a_2, ..., a_n) : a_k \in \{+, -\}, a_k = a_{n-k+1}, 1 \leq k \leq n\}$ be the set of all symmetric n-tuples. A *Smarandachely symmetric n-marked graph* is an ordered pair $S_n = (G, \mu)$, where $G = (V, E)$ is a graph called the *underlying graph* of S_n and $\mu : V \to H_n$ is a function. In this note, we obtain two different characterizations of Smarandachely consistent symmetric n-marked graphs. Also, we obtain some results by introducing special types of complementations.

Key Words: Smarandachely symmetric n-marked graphs, consistency, balance, complementation.

AMS(2000): 05C22

§1. Introduction

For graph theory terminology and notation in this paper we follow the book [2]. All graphs considered here are finite and simple.

A *Smarandachely k-marked graph* is an ordered pair $S = (G, \mu)$ where $G = (V, E)$ is a graph called *underlying graph of* S and $\mu : V \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k)$ is a function, where each $\overline{e}_i \in \{+, -\}$.

Let $n \geq 1$ be an integer. An *n-tuple* $(a_1, a_2, ..., a_n)$ is *symmetric*, if $a_k = a_{n-k+1}, 1 \leq k \leq n$. Let $H_n = \{(a_1, a_2, ..., a_n) : a_k \in \{+, -\}, a_k = a_{n-k+1}, 1 \leq k \leq n\}$ be the set of all symmetric n-tuples. Note that H_n is a group under coordinate wise multiplication, and the order of H_n is 2^m, where $m = \lceil \frac{n}{2} \rceil$. A *Smarandachely symmetric n-marked graph* is an ordered pair $S_n = (G, \mu)$, where $G = (V, E)$ is a graph called the *underlying graph* of S_n and $\mu : V \to H_n$ is a function.

In this paper, by an *n-tuple/n-marked graph* we always mean a symmetric n-tuple /

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Smarandachely symmetric n-marked graph.

An n-tuple $(a_1, a_2, ..., a_n)$ is the *identity n-tuple*, if $a_k = +$, for $1 \leq k \leq n$, otherwise it is a *non-identity n-tuple*. In an n-marked graph $S_n = (G, \mu)$ a vertex labelled with the identity n-tuple is called an *identity vertex*, otherwise it is a *non-identity vertex*. Further, in an n-marked graph $S_n = (G, \mu)$, for any $A \subseteq V(G)$ the *n-tuple* $\mu(A)$ is the product of the n-tuples on the vertices of A.

In [3], the authors defined different notions of balance in an n-marked graph $S_n = (G, \mu)$ as follows:

(i) S_n is *μi-balanced*, if product of n-tuples on each component of S_n is identity n-tuple.

(ii) S_n is *consistent* (*inconsistent*), if product of n-tuples on each cycle of S_n is identity n-tuple (non-identity n-tuple).

(iii) S_n is *balanced*, if every cycle (component) contains an even number of non-identity edges.

Note: (1)A μi-balanced (consistent) n-marked graph need not be balanced and conversely.
(2)A consistent n-marked graph need not be μi-balanced and conversely.

Proposition 1(Characterization of consistent n-marked graphs) *An n-marked graph $S_n = (G, \mu)$ is consistent if, and only if, for each k, $1 \leq k \leq n$, the number of n-tuples in any cycle whose k^{th} co-ordinate is $-$ is even.*

Proof Suppose S_n is consistent and let C be a cycle in S_n with number of n-tuples in any cycle whose k^{th} co-ordinate is $-$ is odd, for some k, $1 \leq k \leq n$. Then, the k^{th} co-ordinate in cycle of n-tuples on the vertices of the cycle C is $-$ and C is inconsistent cycle in S_n. Hence S_n is inconsistent a contradiction.

Converse part follows from the definition of consistent n-marked graphs. □

In [1], Acharya defined trunk on graphs as follows: Given a $u - v$ path $P = (u = u_0, u_1, u_2, ..., u_{m-1}, u_m = v)$ of length $m \geq 2$ in a graph G, the subpath $P' = (u_1, u_2, ..., u_{m-1})$ of P is called a *$u - v$ trunk* or the trunk of P. The following result will give the another characterization of consistent n-marked graph.

Proposition 2 *An n-marked graph $S_n = (G, \mu)$ is consistent if, and only if, for any edge $e = uv$, the n-tuple of the trunk of every $u - v$ path of length ≥ 2 is $\mu(u)\mu(v)$.*

Proof **Necessity:** Suppose $S_n = (G, \mu)$ is consistent. Let $e = uv$ be any edge of S_n and $P = (u = u_0, u_1, u_2, ..., u_{m-1}, u_m = v)$ be any $u - v$ path of length $m \geq 2$ in S_n. Then $C = P \cup \{e\}$ is a cycle in S_n which must have the number of n-tuples whose k^{th} co-ordinate is $-$ is even. Therefore,

$$\mu(P')\mu(u)\mu(v) = \mu(P) = \mu(C) = identity \;\; n - tuple \tag{1}$$

where P' is the trunk of P. Clearly (1), implies that $\mu(P')$ and $\mu(u)\mu(v)$ are equal. Since P was an arbitrarily chosen $u - v$ path of length ≥ 2 and also since the edge e was arbitrary by choice the necessary condition follows.

Sufficiency: Suppose that S_n satisfies the condition stated in the Proposition. We need to show that S_n is consistent. Let $C = (v_1, v_2, ..., v_h, v_1)$ be any cycle in S_n. Consider any edge $e = v_i v_{i+1}$ of C where indices are reduced modulo h. Then by the condition, we have

$$\mu(v_i)\mu(v_{i+1}) = \prod_{j \in \mathbf{h} - \{i, i+1\}} \mu(v_j), \qquad \mathbf{h} = \{1, 2, ..., h\} \tag{2}$$

because the section of P of C, not containing the edge $v_i v_{i+1}$, which is a $v_i - v_{i+1}$ path of length ≥ 2 in S_n satisfies the condition. Equation (2) shows that the number of same non-identity vertices in $\{v_i, v_{i+1}\}$ must be of the even or odd as the number of same non-identity vertices in $V(C) - \{v_i, v_{i+1}\}$. Clearly, this is possible if, and only if, the number of n-tuples cycle C whose k^{th} co-ordinate is $-$ is even if, and only if, C is consistent. Since C was an arbitrarily chosen cycle in S_n, it follows that S_n must be consistent. □

If we take $n = 1$ in the above Proposition, then the following result regarding 1-marked graph (i.e, marked graph).

Corollary 3(B. D. Acharya [1]) *A marked graph $S = (G, \mu)$ is consistent if, and only if, for any edge $e = uv$, the sign of the trunk of every $u - v$ path of length ≥ 2 is $\mu(u)\mu(v)$.*

§2. Complementation

In this section, we investigate the notion of complementation of graphs with multiple signs on their vertices. For any $t \in H_n$, the *t-complement* of $a = (a_1, a_2, ..., a_n)$ is: $a^t = at$. The *reversal* of $a = (a_1, a_2, ..., a_n)$ is: $a^r = (a_n, a_{n-1}, ..., a_1)$. For any $T \subseteq H_n$, and $t \in H_n$, the *t-complement* of T is $T^t = \{a^t : a \in T\}$.

Let $S_n = (G, \mu)$ and $S'_n = (G', \mu')$ be two n-marked graphs. Then S_n is said to be *isomorphic* to S'_n and we write $S_n \cong S'_n$, if there exists a bijection $\phi : V \to V'$ such that if $e = uv$ is an edge in S_n, u and v is labeled by $a = (a_1, a_2, ..., a_n)$ and $a' = (a'_1, a'_2, ..., a'_n)$ respectively, then $\phi(u)\phi(v)$ is an edge in S'_n and $\phi(u)$ and $\phi(v)$ which is labeled by a and a' respectively, and conversely.

For each $t \in H_n$, an n-marked graph $S_n = (G, \mu)$ is *t-self complementary*, if $S_n \cong S^t_n$.

Proposition 4 *For all $t \in H_n$, an n-marked graph $S_n = (G, \mu)$ is t-self complementary if, and only if, S^a_n is t-self complementary, for any $a \in H_n$.*

Proof Suppose S_n is t-self complementary. Then, $S_n \cong S^t_n$. This implies $S^a_n \cong S^{at}_n$.

Conversely, suppose that S^a_n is t-self complementary. Then, $S^a_n \cong (S^a_n)^t$. Since $(S^a_n)^a = S_n$. Hence $S_n \cong (S^{at}_n)^a = S^t_n$. □

Proposition 5 *Let $S_n = (G, \mu)$ be an n-marked graph. Suppose the underlying graph of S_n is bipartite. Then, for any $t \in H_n$, S_n is consistent if, and only if, its t-complement S^t_n is consistent.*

Proof Since S_n is consistent, by Proposition 1, for each $k, 1 \leq k \leq n$, the number of n-tuples on any cycle C in G whose k^{th} co-ordinate is $-$ is even. Also, since G is bipartite,

for each $k, 1 \leq k \leq n$, number of n-tuples on C whose k^{th} co-ordinate is $+$ is also even. This implies that the same thing is true in any t-complement of S_n, where t can be any element of H_n. Hence S_n^t is i-balanced. Similarly, the converse follows, since for each $t \in H_n$, the underlying graph of S_n^t is also bipartite. $\square$

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Some Fixed Point Theorems in Fuzzy n-Normed Spaces

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Abstract: The main purpose of this paper is to study the existence of a fixed points in fuzzy n-normed spaces. we proved our main results, a fixed point theorem for a self mapping and a common fixed point theorem for a pair of weakly compatible mappings on fuzzy n-normed spaces. Also we gave some remarks on fuzzy n-normed spaces.

Key Words: Smarandache space, Pseudo-Euclidean space, fuzzy n-normed spaces, n-seminorm.

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§1. Introduction

A Pseudo-Euclidean space is a particular Smarandache space defined on a Euclidean space $\mathbb{R}^n$such that a straight line passing through a point p may turn an angle $\theta_p \geq 0$. If $\theta_p \geq 0$, then p is called a non-Euclidean point. Otherwise, a Euclidean point. In this paper, normed spaces are considered to be Euclidean, i.e., every point is Euclidean.In [7], S. Gähler introduced n-norms on a linear space. A detailed theory of n-normed linear space can be found in [8,10,12-13]. In [8], H. Gunawan and M. Mashadi gave a simple way to derive an $(n-1)$- norm from the n-norm in such a way that the convergence and completeness in the n-norm is related to those in the derived $(n-1)$-norm. A detailed theory of fuzzy normed linear space can be found in [1,3,4,5,6,9,11]. In [14], A. Narayanan and S. Vijayabalaji have extend n-normed linear space to fuzzy n-normed linear space. In section 2, we quote some basic definitions, and we show that a fuzzy n-norm is closely related to an ascending system of n-seminorms. In section 3, we introduce a locally convex topology in a fuzzy n-normed space. In section 4, we consider finite dimensional fuzzy n-normed linear spaces. In section 5, we give some fixed point theorem in fuzzy $n-$ normed spaces.

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§2. Fuzzy n-norms and ascending families of n-seminorms

Let n be a positive integer, and let X be a real vector space of dimension at least n. We recall the definitions of an n-seminorm and a fuzzy n-norm [14].

Definition 2.1 *A function $(x_1, x_2, \ldots, x_n) \mapsto \|x_1, \ldots, x_n\|$ from X^n to $[0, \infty)$ is called an n-seminorm on X if it has the following four properties:*

(S1) $\|x_1, x_2, \ldots, x_n\| = 0$ *if* $x_1, x_2, \ldots, x_n$ *are linearly dependent;*

(S2) $\|x_1, x_2, \ldots, x_n\|$ *is invariant under any permutation of* $x_1, x_2, \ldots, x_n$*;*

(S3) $\|x_1, \ldots, x_{n-1}, cx_n\| = |c|\|x_1, , \ldots, x_{n-1}, x_n\|$ *for any real* c*;*

(S4) $\|x_1, \ldots, x_{n-1}, y + z\| \leqslant \|x_1, \ldots, x_{n-1}, y\| + \|x_1, \ldots, x_{n-1}, z\|$.

An n-seminorm is called a n-norm if $\|x_1, x_2, \ldots, x_n\| > 0$ whenever $x_1, x_2, \ldots, x_n$ are linearly independent.

Definition 2.1 *A fuzzy subset N of $X^n \times \mathbb{R}$ is called a fuzzy n-norm on X if and only if :*

(F1) *For all* $t \leqslant 0$, $N(x_1, x_2, \ldots, x_n, t) = 0$*;*

(F2) *For all* $t > 0$, $N(x_1, x_2, \ldots, x_n, t) = 1$ *if and only if* $x_1, x_2, \ldots, x_n$ *are linearly dependent;*

(F3) $N(x_1, x_2, \ldots, x_n, t)$ *is invariant under any permutation of* $x_1, x_2, \ldots, x_n$*;*

(F4) *For all* $t > 0$ *and* $c \in \mathbb{R}$, $c \neq 0$,

$$N(x_1, x_2, \ldots, cx_n, t) = N(x_1, x_2, \ldots, x_n, \frac{t}{|c|});$$

(F5) *For all* $s, t \in \mathbb{R}$,

$$N(x_1, \ldots, x_{n-1}, y + z, s + t) \geq \min \{N(x_1, \ldots, x_{n-1}, y, s), N(x_1, \ldots, x_{n-1}z, t)\}.$$

(F6) $N(x_1, x_2, \ldots, x_n, t)$ *is a non-decreasing function of* $t \in \mathbb{R}$ *and*

$$\lim_{t \to \infty} N(x_1, x_2, \ldots, x_n, t) = 1.$$

The following two theorems clarify the relationship between Definitions $2,1$ and 2.2.

Theorem 2.1 *Let N be a fuzzy n-norm on X. As in [14] define for $x_1, x_2, \ldots, x_n \in X$ and $\alpha \in (0, 1)$*

$$\|x_1, x_2, \ldots, x_n\|_\alpha := \inf \{t : N(x_1, x_2, \ldots, x_n, t) \geq \alpha\}. \tag{1}$$

Then the following statements hold.

(A1) *For every* $\alpha \in (0, 1)$, $\|\bullet, \bullet, \ldots, \bullet\|_\alpha$ *is an n-seminorm on* X*;*

(A2) If $0 < \alpha < \beta < 1$ *and* $x_1, \ldots, x_n \in X$ *then*

$$\|x_1, x_2, \ldots, x_n\|_\alpha \leqslant \|x_1, x_2, \ldots, x_n\|_\beta;$$

(A3) If $x_1, x_2, \ldots, x_n \in X$ *are linearly independent then*

$$\lim_{\alpha \to 1-} \|x_1, x_2, \ldots, x_n\|_\alpha = \infty.$$

Proof (A1) and (A2) are shown in [14, Theorem 3.4]. Let $x_1, x_2, \ldots, x_n \in X$ be linearly independent, and $t > 0$ be given. We set $\beta := N(x_1, x_2, \ldots, x_n, t)$. It follows from (F2) that $\beta \in [0, 1)$. Then (F6) shows that, for $\alpha \in (\beta, 1)$,

$$\|x_1, x_2, \ldots, x_n\|_\alpha \geqslant t.$$

This proves (A3). □

We now prove a converse of Theorem 2.1.

Theorem 2.2 *Suppose we are given a family* $\|\bullet, \bullet, \ldots, \bullet\|_\alpha$*,* $\alpha \in (0, 1)$*, of* n*-seminorms on* X *with properties (A2) and (A3). We define*

$$N(x_1, x_2, \ldots, x_n, t) := \inf\{\alpha \in (0, 1) : \|x_1, x_2, \ldots, x_n\|_\alpha \geqslant t\}. \tag{2}$$

where the infimum of the empty set is understood as 1*. Then* N *is a fuzzy* n*-norm on* X*.*

Proof (F1) holds because the values of an n-seminorm are nonnegative.

(F2): Let $t > 0$. If $x_1, \ldots, x_n$ are linearly dependent then $N(x_1, \ldots, x_n, t) = 1$ follows from property (S1) of an n-seminorm. If $x_1, \ldots, x_n$ are linearly independent then $N(x_1, \ldots, x_n, t) < 1$ follows from (A3).

(F3) is a consequence of property (S2) of an n-seminorm.

(F4) is a consequence of property (S3) of an n-seminorm.

(F5): Let $\alpha \in (0, 1)$ satisfy

$$\alpha < \min\{N(x_1, \ldots, x_{n-1}, y, s), N(x_1, \ldots, x_{n-1}, z, s)\}. \tag{3}$$

It follows that $\|x_1, \ldots, x_{n-1}, y\|_\alpha < s$ and $\|x_1, \ldots, x_{n-1}, z\|_\alpha < t$. Then (S4) gives

$$\|x_1, \ldots, x_{n-1}, y + z\|_\alpha < s + t.$$

Using (A2) we find $N(x_1, \ldots, x_{n-1}, y+z, s+t) \geqslant \alpha$ and, since α is arbitrary in (3), (F5) follows.

(F6): Definition 2.2 shows that N is non-decreasing in t. Moreover, $\lim_{t\to\infty} N(x_1, \ldots, x_n, t) = 1$ because seminorms have finite values. □

It is easy to see that Theorems 2.1 and 2.2 establish a one-to-one correspondence between fuzzy n-norms with the additional property that the function $t \mapsto N(x_1, \ldots, x_n, t)$ is left-continuous for all $x_1, x_2, \ldots, x_n$ and families of n-seminorms with properties (A2), (A3) and the additional property that $\alpha \mapsto \|x_1, \ldots, x_n\|_\alpha$ is left-continuous for all $x_1, x_2, \ldots, x_n$.

Example 2.3([14,Example 3.3] Let $\|\bullet, \bullet, \ldots, \bullet\|$ be a n-norm on X. Define $N(x_1, x_2, \ldots, x_n, t)$ $= 0$ if $t \leqslant 0$ and, for $t > 0$,

$$N(x_1, x_2, \ldots, x_n, t) = \frac{t}{t + \|x_1, x_2, \ldots, x_n\|}.$$

Then the seminorms (2.1) are given by

$$\|x_1, x_2, \ldots, x_n\|_\alpha = \frac{\alpha}{1-\alpha}\|x_1, x_2, \ldots, x_n\|.$$

§3. The locally convex topology generated by a fuzzy n-norm

In this section (X, N) is a fuzzy n-normed space, that is, X is real vector space and N is fuzzy n-norm on X. We form the family of n-seminorms $\|\bullet, \bullet, \ldots, \bullet\|_\alpha$, $\alpha \in (0, 1)$, according to Theorem 2.1. This family generates a family $\mathcal{F}$ of seminorms

$$\|x_1, \ldots, x_{n-1}, \bullet\|_\alpha, \quad \text{where } x_1, \ldots, x_{n-1} \in X \text{ and } \alpha \in (0, 1).$$

The family $\mathcal{F}$ generates a locally convex topology on X; see [15, Def. (37.9)], that is, a basis of neighborhoods at the origin is given by

$$\{x \in X : p_i(x) \leqslant \epsilon_i \text{ for } i = 1, 2, \ldots, n\},$$

where $p_i \in \mathcal{F}$ and $\epsilon_i > 0$ for $i = 1, 2 \ldots, n$. We call this the locally convex topology generated by the fuzzy n-norm N.

Theorem 3.1 *The locally convex topology generated by a fuzzy n-norm is Hausdorff.*

Proof Given $x \in X$, $x \neq 0$, choose $x_1, \ldots, x_{n-1} \in X$ such that $x_1, \ldots, x_{n-1}, x$ are linearly independent. By Theorem 2.1(A3) we find $\alpha \in (0, 1)$ such that $\|x_1, \ldots, x_{n-1}, x\|_\alpha > 0$. The desired statement follows; see [15, Theorem 37.21]. □

Some topological notions can be expressed directly in terms of the fuzzy-norm N. For instance, we have the following result on convergence of sequences. We remark that the definition of convergence of sequences in a fuzzy n-normed space as given in [20, Definition 2.2] is meaningless.

Theorem 3.2 *Let $\{x_k\}$ be a sequence in X and $x \in X$. Then $\{x_k\}$ converges to x in the locally convex topology generated by N if and only if*

$$\lim_{k \to \infty} N(a_1, \ldots, a_{n-1}, x_k - x, t) = 1 \tag{4}$$

for all $a_1, \ldots, a_{n-1} \in X$ and all $t > 0$.

Proof Suppose that $\{x_k\}$ converges to x in (X, N). Then, for every $\alpha \in (0, 1)$ and all $a_1, a_2, \ldots, a_{n-1} \in X$, there is K such that, for all $k \geqslant K$, $\|a_1, a_2, \ldots, a_{n-1}, x_k - x\|_\alpha < \epsilon$. The latter implies

$$N(a_1, a_2, \ldots, a_{n-1}, x_k - x, \epsilon) \geqslant \alpha.$$

Since $\alpha \in (0,1)$ and $\epsilon > 0$ are arbitrary we see that (4) holds. The converse is shown in a similar way. □

In a similar way we obtain the following theorem.

Theorem 3.3 *Let $\{x_k\}$ be a sequence in X. Then $\{x_k\}$ is a Cauchy sequence in the locally convex topology generated by N if and only if*

$$\lim_{k,m\to\infty} N(a_1,\dots,a_{n-1},x_k - x_m,t) = 1 \tag{5}$$

for all $a_1,\dots,a_{n-1} \in X$ and all $t > 0$.

It should be noted that the locally convex topology generated by a fuzzy n-norm is not metrizable, in general. Therefore, in many cases it will be necessary to consider nets $\{x_i\}$ in place of sequences. Of course, Theorems 3.2 and 3.3 generalize in an obvious way to nets.

§4. Fuzzy n-norms on finite dimensional spaces

In this section (X,N) is a fuzzy n-normed space and X has finite dimension at least n. Since the locally convex topology generated by N is Hausdorff by Theorem 3.1 Tihonov's theorem [15, Theorem 23.1] implies that this locally convex topology is the only one on X. Therefore, all fuzzy n-norms on X are equivalent in the sense that they generate the same locally convex topology.

In the rest of this section we will give a direct proof of this fact (without using Tihonov's theorem). We will set $X = \mathbb{R}^d$ with $d \geqslant n$.

Lemma 4.1 *Every n-seminorm on $X = \mathbb{R}^d$ is continuous as a function on X^n with the euclidian topology.*

Proof For every $j = 1,2,\dots,n$, let $\{x_{j,k}\}_{k=1}^{\infty}$ be a sequence in X converging to $x_j \in X$. Therefore, $\lim\limits_{k\to\infty} \|x_{j,k} - x_j\| = 0$, where $\|x\|$ denotes the euclidian norm of x. From property (S4) of an n-seminorm we get

$$|\|x_{1,k},x_{2,k},\dots,x_{n,k}\| - \|x_1,x_{2,k},\dots,x_{n,k}\|| \le \|x_{1,k}-x_1,x_{2,k},\dots,x_{n,k}\|.$$

Expressing every vector in the standard basis of $\mathbb{R}^d$ we see that there is a constant M such that

$$\|y_1,y_2,\dots,\ y_n\| \le M\,\|y_1\|\dots\|y_n\| \ \text{ for all } y_j \in X.$$

Therefore,

$$\lim_{k\to\infty}\ \|x_{1,k}-x_1,\ x_{2,k},\dots,\ x_{n,k}\| = 0$$

and so

$$\lim_{k\to\infty}\ |\|x_{1,k},\ x_{2,k},\dots,\ x_{n,k}\| - \|x_1,\ x_{2,k},\dots,\ x_{n,k}\|| = 0.$$

We continue this procedure until we reach

$$\lim_{k\to\infty} \|x_{1,k},\ x_{2,k},\dots,\ x_{n,k}\| = \|x_1,\ x_2,\dots,\ x_n\|\,. \qquad \square$$

Lemma 4.2 *Let $(\mathbb{R}^d, N)$ be a fuzzy n-normed space. Then $\|x_1, x_2, \ldots, x_n\|_\alpha$ is an n-norm if $\alpha \in (0,1)$ is sufficiently close to 1.*

Proof We consider the compact set

$$S = \left\{(x_1,\ x_2, \ldots,\ x_n) \in \mathbb{R}^{dn} : x_1,\ x_2, \ldots,\ x_n \text{ is an orthonormal system in } \mathbb{R}^d\right\}.$$

For each $\alpha \in (0,1)$ consider the set

$$S_\alpha = \{(x_1, x_2, \ldots, x_n) \in S : \|x_1, x_2, \ldots, x_n\|_\alpha > 0\}.$$

By Lemma 4.1, S_α is an open subset of S. We now show that

$$S = \bigcup_{\alpha \in (0,1)} S_\alpha. \tag{6}$$

If $(x_1,\ x_2, \ldots,\ x_n) \in S$ then $(x_1,\ x_2, \ldots,\ x_n)$ is linearly independent and therefore there is β such that $N(x_1, x_2, \ldots, x_n, 1) < \beta < 1$. This implies that $\|x_1,\ x_2, \ldots,\ x_n\|_\beta \geq 1$ so (6) is proved. By compactness of S, we find $\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_m$ such that

$$S = \bigcup_{i=1}^{m} S_{\alpha_i}.$$

Let $\alpha = \max\{\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_m\}$. Then $\|x_1,\ x_2, \ldots,\ x_n\|_\alpha > 0$ for every $(x_1,\ x_2, \ldots,\ x_n) \in S$.

Let $x_1,\ x_2, \ldots,\ x_n \in X$ be linearly independent. Construct an orthonormal system $e_1,\ e_2, \ldots,\ e_n$ from $x_1,\ x_2, \ldots,\ x_n$ by the Gram-Schmidt method. Then there is $c > 0$ such that

$$\|x_1,\ x_2, \ldots,\ x_n\|_\alpha = c\,\|e_1,\ e_2, \ldots,\ e_n\|_\alpha > 0.$$

This proves the lemma. □

Theorem 4.1 *Let N be a fuzzy n-norm on $\mathbb{R}^d$, and let $\{x_k\}$ be a sequence in $\mathbb{R}^d$ and $x \in \mathbb{R}^d$.*

(a) $\{x_k\}$ converges to x with respect to N if and only if $\{x_k\}$ converges to x in the euclidian topology.

(b) $\{x_k\}$ is a Cauchy sequence with respect to N if and only if $\{x_k\}$ is a Cauchy sequence in the euclidian metric.

Proof (a) Suppose $\{x_k\}$ converges to x with respect to euclidian topology. Let $a_1, a_2, \ldots, a_{n-1} \in X$. By Lemma 4.1, for every $\alpha \in (0,1)$,

$$\lim_{k\to\infty} \|a_1,\ a_2, \ldots,\ a_{n-1},\ x_k - x\|_\alpha = 0.$$

By definition of convergence in $(\mathbb{R}^d, N)$, we get that $\{x_k\}$ converges to x in $(\mathbb{R}^d, N)$. Conversely, suppose that $\{x_k\}$ converges to x in $(\mathbb{R}^d, N)$. By Lemma 4.2, there is $\alpha \in (0,1)$ such that $\|y_1, y_2, \ldots, y_n\|_\alpha$ is an n-norm. By definition, $\{x_k\}$ converges to x in the n-normed space $(\mathbb{R}^d, \|\cdot\|_\alpha)$. It is known from[8, Proposition 3.1] that this implies that $\{x_k\}$ converges to x with respect to euclidian topology.

(b) is proved in a similar way. □

Theorem 4.2 *A finite dimensional fuzzy n-normed space (X, N) is complete.*

Proof This follows directly from Theorem 3.4. $\square$

§5. Some fixed point theorem in fuzzy $n-$ normed spaces

In this section we prove some fixed point theorems.

Definition 5.1 *A sequence a $\{x_k\}$ in a fuzzy n-normed space (X, N) is said to be fuzzy n-convergent to $x^* \in X$ and denoted by $x_k \rightsquigarrow x^*$ as $k \to \infty$ if*

$$\lim_{k\to\infty} N(x_1, \cdots, x_{n-1}, x_k - x^*, t) = 1$$

for every $x_1, \cdots, x_{n-1} \in X$ and x^* is called the fuzzy n-limit of $\{x_k\}$.

Remark 5.1 It is noted that if (X, N) is a fuzzy n-normed space then the fuzzy n-limit of a fuzzy n-convergent sequence is unique. Indeed, if $\{x_k\}$ is a fuzzy n-convergent sequence and suppose it converges to x^* and y^* in X. Then by definition $\lim_{k\to\infty} N(x_1, \cdots, x_{n-1}, x_k - x^*, t) = 1$ and $\lim_{k\to\infty} N(x_1, \cdots, x_{n-1}, x_k - y^*, t) = 1$ for every $x_1, \cdots, x_{n-1} \in X$ and for every $t > 0$. By $(N5)$, we have

$$\begin{aligned} N(x_1, \cdots, x_{n-1}, x - y, t) &= N(x_1, \cdots, x_{n-1}, x^* - x_k + x_k - y^*, t/2 + t/2) \\ &\geqslant \min\{N(x_1, \cdots, x_{n-1}, x^* - x_k, t/2), N(x_1, \cdots, x_{n-1}, x_k - y^*, t/2)\}. \end{aligned}$$

By letting $k \to \infty$, we obtain $N(x_1, \cdots, x_{n-1}, x^* - y^*, t) = 1$, which implies that $x^* = y^*$.

Definition 5.2 *A sequence $\{x_k\}$ in a fuzzy n-normed space (X, N) is said to be fuzzy n-Cauchy sequence if*

$$\lim_{k,m\to\infty} N(x_1, \ldots, x_{n-1}, x_k - x_m, t) = 1$$

for every $x_1, \cdots, x_{n-1} \in X$ and for every $t > 0$.

Proposition 5.1 *In a fuzzy n-normed space (X, N), every fuzzy n-convergent sequence is a fuzzy n-Cauchy sequence.*

Proof Let $\{x_k\}$ be a fuzzy n-convergent sequence in X converging to $x^* \in X$. Then $\lim_{k\to\infty} N(x_1, \cdots, x_{n-1}, x_k - x^*, t) = 1$ for every $x_1, \cdots, x_{n-1} \in X$ and for every $t > 0$. By (N5),

$$\begin{aligned} &N(x_1, \cdots, x_{n-1}, x_k - x_m, t) \\ &\quad = N(x_1, \cdots, x_{n-1}, x_k - x^* + x^* - x_m, t/2 + t/2) \\ &\quad \geqslant \min\{N(x_1, \cdots, x_{n-1}, x_k - x^*, t/2), N(x_1, \cdots, x_{n-1}, x^* - x_m, t/2)\}. \end{aligned}$$

By letting $n, m \to \infty$, we get,

$$\lim_{k,m\to\infty} N(x_1, \cdots, x_{n-1}, x_k - x_m, t) = 1$$

for every $x_1, \cdots, x_{n-1} \in X$ and for every $t > 0$, i.e., $\{x_k\}$ is a fuzzy n-Cauchy sequence. $\square$

If every fuzzy n-Cauchy sequence in X converges to an $x^* \in X$, then (X, N) is called a complete fuzzy n-normed space. A complete fuzzy n-normed space is then called a fuzzy n-Banach space.

Theorem 5.1 *Let (X, N) be a fuzzy n-normed space. Let $f : X \to X$ be a map satisfies the condition:*
There exists a $\lambda \in (0, 1)$ such that for all $x, x_1, \cdots, x_{n-1} \in X$ and for all $t > 0$, one has

$$N(x_1, \cdots, x_{n-1}, x, t) > 1 - t \quad \Rightarrow \quad N(x_1, \cdots, x_{n-1}, f(x), \lambda t) > 1 - \lambda t. \tag{7}$$

Then

(i) For any real number $\epsilon > 0$ there exists $k_0(\epsilon) \in \mathbb{N}$ such that $f^k(x) \rightsquigarrow \theta$.

(ii) f has at most a fixed point, that is the null vector of X. Moreover, if f is a linear mapping, f has exactly one fixed point.

Proof (i) Note that if f satisfies the condition (1), then for every $\epsilon \in (0, 1)$, there exists a $k_0 = k_0(\epsilon)$ such that, for all $k \geqslant k_0$, and for every $x, x_1, \cdots, x_{n-1} \in X$

$$N(x_1, \cdots, x_{n-1}, f^k(x), \epsilon) > 1 - \epsilon$$

holds. Indeed, one has easily that

$$N(x_1, \cdots, x_{n-1}, x, 1 + \epsilon) > 1 - (1 + \epsilon).$$

Then by condition (1), for all $x, x_1, \cdots, x_{n-1} \in X$ and $k \geqslant 1$,

$$N(x_1, \cdots, x_{n-1}, f^k(x), \lambda^k(1 + \epsilon)) > 1 - \lambda^k(1 + \epsilon)$$

holds. Indeed, for each $\epsilon > 0$ there exists a $k = k_0$ implies that $\lambda^n(1 + \epsilon) \leqslant \epsilon$, from which, because of condition (N6), there exists a $k_0 \in \mathbb{N}$ such that for $k \geqslant k_0$,

$$\begin{aligned} N(x_1, \cdots, x_{n-1}, f^k(x), \epsilon) &\geqslant N(x_1, \cdots, x_{n-1}, f^k(x), \lambda^k(1 + \epsilon)) \\ &> 1 - \lambda^k(1 + \epsilon) \\ &\geqslant 1 - \epsilon. \end{aligned}$$

Since ϵ is an arbitrary, we have $f^k(x) \rightsquigarrow \theta$ as required.

(ii) Assume that $f(x) = x$. By applying part (i), for all $\epsilon \in (0, 1)$ one has

$$N(x_1, \cdots, x_{n-1}, x, \epsilon) > 1 - \epsilon$$

for every $x_1, \cdots, x_{n-1} \in X$. This implies that

$$N(x_1, \cdots, x_{n-1}, x, 0+) = 1$$

for every $x_1, \cdots, x_{n-1} \in X$, i.e., $x = \theta$. $\square$

Lemma 5.1 *Let $\{x_k\}$ be a sequence in a fuzzy n-normed space (X, M). If for every $t > 0$, there exists a constant $\lambda \in (0,1)$ such that*

$$N(x_1, \ldots, x_{n-1}, x_k - x_{k+1}, t) \geq N(x_1, \ldots, x_{n-1}, x_{k-1} - x_k, t/\lambda) \tag{8}$$

for all $x_1, \cdots, x_{n-1} \in X$, then $\{x_k\}$ is a fuzzy n-Cauchy sequence in X.

Proof Let $t > 0$ and $\lambda \in (0,1)$. Then for $m \geqslant k$, by using (N5) and the inequality (1), we have

$$\begin{aligned}
&N(x_1, \ldots, x_{n-1}, x_k - x_m, t)\\
&\quad \geqslant \min\{N(x_1, \ldots, x_{n-1}, x_k - x_{k+1}, (1-\lambda)t),\\
&\qquad N(x_1, \ldots, x_{n-1}, x_{k+1} - x_m, \lambda t)\}\\
&\qquad \ldots\\
&\quad \geqslant \min\{N(x_1, \ldots, x_{n-1}, x_0 - x_1, \frac{(1-\lambda)t}{\lambda^k}),\\
&\qquad N(x_1, \ldots, x_{n-1}, x_{k+1} - x_m, \lambda t)\}
\end{aligned}$$

Also,

$$\begin{aligned}
&N(x_1, \ldots, x_{n-1}, x_{k+1} - x_m, \lambda t)\\
&\quad \geqslant \min\{N(x_1, \ldots, x_{n-1}, x_{k+1} - x_{k+2}, (1-\lambda)\lambda t),\\
&\qquad N(x_1, \ldots, x_{n-1}, x_{k+2} - x_m, \lambda^2 t)\}\\
&\qquad \ldots\\
&\quad \geqslant \min\{N(x_1, \ldots, x_{n-1}, x_0 - x_1, \frac{(1-\lambda)t}{\lambda^k}),\\
&\qquad N(x_1, \ldots, x_{n-1}, x_{k+2} - x_m, \lambda^2 t)\}
\end{aligned}$$

By repeating these argument, we get

$$\begin{aligned}
&N(x_1, \ldots, x_{n-1}, x_k - x_m, t)\\
&\quad \geqslant \min\{N(x_1, \ldots, x_{n-1}, x_0 - x_1, \frac{(1-\lambda)t}{\lambda^k}),\\
&\qquad N(x_1, \ldots, x_{n-1}, x_{m-1} - x_m, \lambda^{m-n-1} t)\}\\
&\qquad \ldots\\
&\quad \geqslant \min\{N(x_1, \ldots, x_{n-1}, x_0 - x_1, \frac{(1-\lambda)t}{\lambda^k}),\\
&\qquad N(x_1, \ldots, x_{n-1}, x_0 - x_1, \frac{t}{\lambda^k})\}
\end{aligned}$$

Since $(1-\lambda)\frac{t}{\lambda^k} \leq \frac{t}{\lambda^k}$ and the property (F6), we conclude that

$$N(x_1, \ldots, x_{n-1}, x_k - x_m, t) \geq N(x_1, \ldots, x_{n-1}, x_0 - x_1, \frac{(1-\lambda)t}{\lambda^k}).$$

Therefore, by letting $m \geqslant k \to \infty$, we get

$$\lim_{k,m\to\infty} N(x_1, \cdots, x_{n-1}, x_k - x_m, t) = 1$$

for every $x_1, \cdots, x_{n-1} \in X$ and for every $t > 0$, i.e., $\{x_k\}$ is a fuzzy n-Cauchy sequence. □

Definition 5.3 *A pair of maps (f, g) is called weakly compatible pair if they commute at coincidence point, i.e., $fx = gx$ implies $fgx = gfx$.*

Theorem 5.2 *Let (X, M) be a fuzzy n-normed space and let $f, g\colon X \to X$ satisfy the following conditions:*

(i) $f(X) \subseteq g(X)$;

(ii) *any one* $f(X)$ *or* $g(X)$ *is complete;*

(iii) $N(x_1, \ldots, x_{n-1}, f(x)-f(y), t) \geqslant N(x_1, \ldots, x_{n-1}, g(x)-g(y), t/\lambda)$, *for all* $x, y, x_1, \cdots, x_{n-1} \in X$, $t > 0$, $\lambda \in (0, 1)$.

Then f and g have a unique common fixed point provided f and g are weakly compatible on X.

Proof Let $x_0 \in X$. By condition (i), we can find $x_1 \in X$ such that $f(x_0) = g(x_1) = y_1$. By induction, we can define a sequence y_k in X such that

$$y_{k+1} = f(x_k) = g(x_{k+1}),$$

$n = 0, 1, 2, \cdots$. We consider two cases:
Case I: If $y_r = y_{r+1}$ for some $r \in \mathbb{N}$, then

$$y_r = f(x_{r-1}) = f(x_r) = g(x_r) = g(x_{r+1}) = y_{r+1} = z$$

for some $z \in X$. Since $f(x_r) = g(x_r)$ and f, g are weakly compatible, we have $f(z) = fg(x_r) = gf(x_r) = g(z)$. By condition (iii), for all $x_1, \cdots, x_{n-1} \in X$ and for all $t > 0$, we have

$$\begin{aligned} N(x_1, \cdots, x_{n-1}, f(z) - z, t) &= N(x_1, \cdots, x_{n-1}, f(z) - f(x_r), t) \\ &\geqslant N(x_1, \cdots, x_{n-1}, g(z) - g(x_r), t/\lambda) \\ &\geqslant \cdots \geq N(x_1, \cdots, x_{n-1}, g(z) - g(x_r), t/\lambda^k). \end{aligned}$$

Clearly, the righthand side of the inequality approaches 1 as $k \to \infty$ for every $x_1, \ldots, x_{n-1} \in X$ and $t > 0$. Hence, $N(x_1, \cdots, x_{n-1}, f(z) - z, t) = 1$. This implies that $f(z) = z = g(z)$, i.e., z is a common fixed point of f and g.
Case II $y_k \neq y_{k+1}$, for each $k = 0, 1, 2, \cdots$. Then, by condition (ii) again, we have

$$\begin{aligned} N(x_1, \cdots, x_{n-1}, y_k - y_{k+1}, t) &= N(x_1, \cdots, x_{n-1}, g(x_k) - g(x_{k+1}), t) \\ &= N(x_1, \cdots, x_{n-1}, f(x_{k-1}) - f(x_k), t) \\ &\geq N(x_1, \cdots, x_{n-1}, g(x_{k-1}) - g(x_k), t/\lambda) \\ &= N(x_1, \cdots, x_{n-1}, y_{k-1} - y_k, t) \end{aligned}$$

Then, by Lemma 5.1, $\{y_k\}$ is a Cauchy sequence (with respect to fuzzy n-norm) in X. Since $g(X)$ is complete, there exists $w \in g(X)$ such that

$$\lim_{k\to\infty} y_k = \lim_{k\to\infty} g(x_k) = w.$$

Also, since $w \in g(X)$, we can find a $p \in X$ such that $g(p) = w$. Note that

$$w = g(p) = \lim_{k\to\infty} g(x_k) = \lim_{k\to\infty} f(x_k).$$

Thus, by (iii), we have

$$\begin{aligned}
N(x_1, \cdots, x_{n-1}, f(p) - g(p), t) &= \lim_{k\to\infty} N(x_1, \cdots, x_{n-1}, f(p) - f(x_k), t) \\
&\geq \lim_{k\to\infty} N(x_1, \cdots, x_{n-1}, g(p) - g(x_k), t/\lambda) \\
&= N(x_1, \cdots, x_{n-1}, g(p) - w, t/\lambda) \\
&= N(x_1, \cdots, x_{n-1}, w - w, t/\lambda),
\end{aligned}$$

which implies that $w = f(p) = g(p)$ is a common fixed point of f and g. Furthermore, f and g are weakly compatible maps, we have

$$f(w) = fg(w) = gf(w) = g(w).$$

But than, by (iii),

$$\begin{aligned}
N(x_1, \cdots, x_{n-1}, f(w) - w, t) &= N(x_1, \cdots, x_{n-1}, f(w) - f(p), t) \\
&\geq N(x_1, \cdots, x_{n-1}, g(w) - g(p), t/\lambda) \\
&= N(x_1, \cdots, x_{n-1}, f(w) - f(p), t/\lambda) \\
&\geq \cdots \geq N(x_1, \cdots, x_{n-1}, g(w) - g(p), t/\lambda^k).
\end{aligned}$$

Clearly, the expression on the righthand side approaches 1 as $k \to \infty$ for every $x_1, \ldots, x_{n-1} \in X$ and $t > 0$, which implies that $f(w) = w$. Therefore, w is a common fixed point of f and g. The uniqueness of fixed point is immediate from condition (iii). $\square$

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A Result of Ramanujan and Brahmagupta Polynomials Described by a Matrix Identity

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Abstract: In the present paper, the following result of Ramanujan [2] is shown to be contained as special case of a matrix identity in two parameters [3]: If a , b , c , d are real numbers such that $a\,d \;-\; b\,c \;=\; 0$, then

$$(a + b + c)^2 + (b + c + d)^2 + (a - d)^2 = (c + d + a)^2 + (d + a + b)^2 + (b - c)^2.$$

$$(a + b + c)^4 + (b + c + d)^4 + (a - d)^4 = (c + d + a)^4 + (d + a + b)^4 + (b - c)^4.$$

Combinatorial properties of the two pairs of Brahmagupta polynomials defined by the matrix identities in one and two parameters are also described.

Key Words: Results of Ramanujan, matrix identity, Brahmagupta polynomials, combinatorial properties.

AMS(2000): 01A32, 11B37, 11B39

§1. Introduction

E.R. Suryanarayan [4] has described the following matrix identity:

$$\begin{bmatrix} x_n & y_n \\ t\,y_n & x_n \end{bmatrix} = \begin{bmatrix} x & y \\ t\,y & x \end{bmatrix}^n \tag{1}$$

with $x_0 \;=\; 1,\; y_0 \;=\; 0,\; n = 0, 1, 2, \cdots$. The identity (1) is the starting point to define a pair of homogeneous polynomials $\{x_n(x, y; t), y_n(x, y; t)\}$ of degree n in two real variables x, y and a real parameter $t \neq 0$ such that $x^2 \;-\; ty^2 \neq 0$ called Brahmagupta Polynomials. An extensive list of properties of Brahmagupta polynomials is given in [4].

R.Rangarajan, Rangaswamy and E.R. Suryanarayan [3] have extended the matrix identity (1) in the following way: Let $\mathbf{B}^{(\mathbf{s},\mathbf{t})}$ denote the set of matrices of the form

$$B = \begin{bmatrix} x & y \\ ty & x + sy \end{bmatrix} \tag{2}$$

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where t and s are two parameters and x and y are two real variables subjected to the condition that $x^2 + s\,x\,y - t\,y^2 \neq 0$. Define B to be the extended matrix in two parameters. It is easy to check that in $\mathbf{B^{(s,t)}}$ the commutative law for multiplication holds. As a result,the following extended matrix identity in two parameters holds:

$$\begin{bmatrix} x & y \\ ty & x+sy \end{bmatrix}^n = \begin{bmatrix} x_n(x,y,s,t) & y_n(x,y,s,t) \\ ty_n(x,y,s,t) & x_n(x,y,s,t)+sy_n(x,y,s,t) \end{bmatrix} \tag{3}$$

It is very interesting to note that, if $s=t=y=1$ and $x=0$, then (3) takes the form:

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n = \begin{bmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{bmatrix} \tag{4}$$

where F_n is the n^{th} Fibonacci number

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

The extended matrix identity (3) defines the pair $(x_n(x,y,s,t)\ ,\ y_n(x,y,s,t))$ of Brahmagupta polynomials in two parameters. An extensive list of properties of Brahmagupta polynomials in two parameters is given in [3].

In [1] an innovative matrix identity wherein each matrix has a determinant of the form $x^2+y^2+z^2$ is proposed to view Ramanujan result in the power 2. But the identity does not work in the power 4. However, the paper provided us a good motivation to seek an appropriate matrix identity in two parameters to view both the results of Ramanujan.

§2. A pair of results of Ramanujan

One of the remarkable results of Ramanujan, appearing on the page 385 of his note books [2] is stated as follows: If a,b,c,d are real numbers such that $ad=bc$, then

$$(a+b+c)^2+(b+c+d)^2+(a-d)^2=(c+d+a)^2+(d+a+b)^2+(b-c)^2 \tag{5}$$

$$(a+b+c)^4+(b+c+d)^4+(a-d)^4=(c+d+a)^4+(d+a+b)^4+(b-c)^4 \tag{6}$$

For example, if $a=6, b=3, c=2$ and $d=1$, then $11^2+6^2+5^2=9^2+10^2+1^2$ and $11^4+6^4+5^4=9^4+10^4+1^4$. Writing

$$x_1=a+b+c,\ y_1=b+c+d,\ z_1=c+d+a,\ w_1=d+a+b$$

the results (5) and (6) become

$$x_1^2+y_1^2+(x_1-y_1)^2=z_1^2+w_1^2+(z_1-w_1)^2 \tag{7}$$

$$x_1^4+y_1^4+(x_1-y_1)^4=z_1^4+w_1^4+(z_1-w_1)^4 \tag{8}$$

where x_1, y_1, z_1, w_1 are real numbers such that $x_1^2 + y_1^2 - x_1 y_1 = z_1^2 + w_1^2 - z_1 w_1$.
It is straightforward to workout

$$\begin{aligned} a &= \frac{1}{3}\ x_1 - \frac{2}{3}\ y_1 + \frac{1}{3}\ z_1 + \frac{1}{3}\ w_1, \\ b &= \frac{1}{3}\ x_1 + \frac{1}{3}\ y_1 - \frac{2}{3}\ z_1 + \frac{1}{3}\ w_1, \\ c &= \frac{1}{3}\ x_1 + \frac{1}{3}\ y_1 + \frac{1}{3}\ z_1 - \frac{2}{3}\ w_1, \\ d &= -\frac{2}{3}\ x_1 + \frac{1}{3}\ y_1 + \frac{1}{3}\ z_1 + \frac{1}{3}\ w_1 \end{aligned}$$

and hence $ad = bc$ is equivalent to

$$x_1^2 + y_1^2 - x_1 y_1 = z_1^2 + w_1^2 - z_1 w_1.$$

Now, it is very easy to verify the Ramanujan results because on expanding the last terms and simplifying both the sides of (7) and (8) one obtains:

$$2(x_1^2 + y_1^2 - x_1 y_1) = 2(z_1^2 + w_1^2 - z_1 w_1) \tag{9}$$

$$2(x_1^2 + y_1^2 - x_1 y_1)^2 = 2(z_1^2 + w_1^2 - z_1 w_1)^2 \tag{10}$$

By varying the choices for a, b, c, d one obtains infinitely many solutions of (5) and (6).
The main purpose of this paper is to generate infinite quadruple sequences of solutions $\{x_n, y_n, z_n, w_n\}, n = 1, 2, 3, \cdots$ to (7) and (8) starting from just one set $\{x_1, y_1, z_1, w_1\}$ of positive integers such that $x_n^2 + y_n^2 - x_n y_n = z_n^2 + w_n^2 - z_n w_n \neq 0$, using a suitable extended matrix in two parameters (2) wherein each matrix has a determinant of the form

$$x_1^2 + y_1^2 - x_1 y_1 = \frac{1}{2}(x_1^2 + y_1^2 + (x_1 - y_1)^2).$$

This new idea enables us to construct a pair of two variable homogeneous polynomials of degree n which are useful to evaluate $\{x_n, y_n, z_n, w_n\}, n = 1, 2, 3, \cdots$.

The required extended matrix identity in two parameters: In order to achieve our objective, we shall consider the set of all the matrices appearing in the identity (3) with $s = t = -1$:

$$A(x, y) = \begin{pmatrix} x & y \\ -y & x - y \end{pmatrix} \tag{11}$$

where x and y are any two real numbers such that $x^2 + y^2 - xy \neq 0$. Clearly $A(x, y) \in GL_2(\mathbb{R})$, general linear group of all 2 by 2 invertible matrices. Let $\mathbb{A}_{(x,y)}$ be the set of all matrices of the form (11) where x and y are any two real numbers such that $x^2 + y^2 - xy \neq 0$.
Let $A(x_1, y_1)$ and $A(x_2, y_2)$ be any two matrices in $\mathbb{A}_{(x,y)}$. Then we shall show that $A(x_3, y_3) = A(x_1, y_1)A(x_2, y_2)$ is also in $\mathbb{A}_{(x,y)}$.

$$A(x_3, y_3) = \begin{pmatrix} x_1 & y_1 \\ -y_1 & x_1 - y_1 \end{pmatrix} \begin{pmatrix} x_2 & y_2 \\ -y_2 & x_2 - y_2 \end{pmatrix}$$

$$= \begin{pmatrix} (x_1x_2 - y_1y_2) & (x_1y_2 + y_1x_2 - y_1y_2) \\ -(x_1y_2 + y_1x_2 - y_1y_2) & (x_1x_2 - y_1y_2) - (x_1y_2 + y_1x_2 - y_1y_2) \end{pmatrix}$$

where $x_3 = x_1x_2 - y_1y_2$ and $y_3 = (x_1y_2 + y_1x_2 - y_1y_2)$ are again real numbers and $x_3^2 + y_3^2 - x_3y_3 = (x_1^2 + y_1^2 - x_1y_1)(x_2^2 + y_2^2 - x_2y_2) \neq 0$. Moreover,

$$A(x_1, y_1)A(x_2, y_2) = A(x_2, y_2)A(x_1, y_1).$$

Hence $\mathbb{A}_{(x,y)}$ is a commutative matrix subgroup of $GL_2(\mathbb{R})$. In this matrix subgroup, Ramanujan result deduced in (9) and (10) can be restated as follows:

$$2det[A(x_1, y_1)] = 2 \quad det[A(z_1, w_1)] \tag{12}$$

$$2\{det[A(x_1, y_1)]\}^2 = 2 \quad \{det[A(z_1, w_1)]\}^2 \tag{13}$$

Now, the infinite quadruple solutions $\{x_n, y_n, z_n, w_n\}, n = 1, 2, 3, \cdots$ can be computed as follows:

$$A(x_n, y_n) = [A(x_1, y_1)]^n \tag{14}$$

$$A(z_n, w_n) = [A(z_1, w_1)]^n \tag{15}$$

Using the standard theorem on product of determinants, it is straight forward to workout

$$2 \quad det[A(x_n, y_n)] = 2 \quad det[A(z_n, w_n)] \tag{16}$$

$$2 \quad \{det[A(x_n, y_n)]\}^2 = 2 \quad \{det[A(z_n, w_n)]\}^2 \tag{17}$$

In order to workout (14) and (15), we shall use the following eigen relations:

$$\begin{pmatrix} x & y \\ -y & x - y \end{pmatrix}^n = \frac{1}{\omega^2 - \omega} \begin{pmatrix} 1 & 1 \\ \omega & \omega^2 \end{pmatrix} \begin{pmatrix} x + \omega y & 0 \\ 0 & x + \omega^2 y \end{pmatrix}^n \begin{pmatrix} \omega^2 & -1 \\ -\omega & 1 \end{pmatrix}$$

where $\omega = e^{\frac{2\pi i}{3}}$ is the cube root of unity. As a result, $\{x_n, y_n, z_n, w_n\}, n = 1, 2, 3, \cdots$ have the following binet forms:

$$x_n = \frac{-\omega^2(x_1 + \omega y_1)^n + \omega(x_1 + \omega^2 y_1)^n}{\omega - \omega^2} \tag{18}$$

$$y_n = \frac{(x_1 + \omega y_1)^n - (x_1 + \omega^2 y_1)^n}{\omega - \omega^2} \tag{19}$$

$$z_n = \frac{-\omega^2(z_1 + \omega w_1)^n + \omega(z_1 + \omega^2 w_1)^n}{\omega - \omega^2} \tag{20}$$

$$w_n = \frac{(z_1 + \omega w_1)^n - (z_1 + \omega^2 w_1)^n}{\omega - \omega^2} \tag{21}$$

Also, it is interesting to workout the following binary recurrence relations for $\{x_n, y_n, z_n, w_n\}, n = 1, 2, 3, \cdots$:

$$x_{n+1} = (2x_1 - y_1) \quad x_n - (x_1^2 + y_1^2 - x_1y_1) \quad x_{n-1}, x_0 = 1, x_1 = a + b + c \tag{22}$$

$$y_{n+1} = (2x_1 - y_1) \quad y_n - (x_1^2 + y_1^2 - x_1y_1) \quad y_{n-1}, y_0 = 0, y_1 = b + c + d \tag{23}$$

$$z_{n+1} = (2z_1 - w_1)\ \ z_n - (z_1^2 + w_1^2 - z_1 w_1)\ \ z_{n-1}, z_0 = 1, z_1 = c + d + a \tag{24}$$

$$w_{n+1} = (2z_1 - w_1)\ \ w_n - (z_1^2 + w_1^2 - z_1 w_1)\ \ w_{n-1}, w_0 = 0, w_1 = d + a + b \tag{25}$$

where a, b, c, d are any four real numbers such that $ad = bc$.

A pair of evaluating polynomials: The binet forms (18) – (21) define a Pair of Evaluating Polynomials,namely, $P_n(x, y)$ and $Q_n(x, y)$ given by

$$P_n(x, y) = \frac{-\omega^2(x + \omega y)^n + \omega(x + \omega^2 y)^n}{\omega - \omega^2} \tag{26}$$

$$Q_n(x, y) = \frac{(x + \omega y)^n - (x + \omega^2 y)^n}{\omega - \omega^2} \tag{27}$$

So that one can evaluate

$$P_n(x_1, y_1) = x_n, Q_n(x_1, y_1) = y_n, P_n(z_1, w_1) = z_n, Q_n(z_1, w_1) = w_n.$$

It is also a quite convenient method for computing $(P_n(x, y), Q_n(x, y))$ using the following extended matrix identity:

$$\begin{pmatrix} P_n(x, y) & Q_n(x, y) \\ -Q_n(x, y) & P_n(x, y) - Q_n(x, y) \end{pmatrix} = \begin{pmatrix} x & y \\ -y & x - y \end{pmatrix}^n$$

§3. Combinatorial properties of Brahmagupta Polynomials

The Brahmagupta polynomials in one parameter exhibit the following combinatorial properties:

Theorem 1([4]) *The Brahmagupta polynomials in one parameter have the following binet forms :*

$$\left.\begin{aligned} x_n &= \frac{1}{2}\left[(x + y\sqrt{t})^n + (x - y\sqrt{t})^n\right] \\ y_n &= \frac{1}{2\sqrt{t}}\left[(x + y\sqrt{t})^n - (x - y\sqrt{t})^n\right] \end{aligned}\right\}. \tag{28}$$

They satisfy the following three -term recurrences :

$$\left.\begin{aligned} x_{n+1} &= 2\,x\,x_n - (x^2 - ty^2)\,x_{n-1}\,,\ x_0 = 1,\ x_1 = x \\ y_{n+1} &= 2\,x\,y_n - (x^2 - ty^2)\,y_{n-1}\,,\ y_0 = 0,\ y_1 = y \end{aligned}\right\}. \tag{29}$$

The Brahmagupta polynomials in two parameters exhibit the following similar combinatorial properties:

Theorem 2([3]) $\left(x_n + \dfrac{s}{2} y_n\right)$ *and* y_n *have the following binet forms:*

$$\left.\begin{aligned} \left(x_n + \tfrac{s}{2} y_n\right) &= \tfrac{1}{2}\left[(x+\lambda_+ y)^n + (x+\lambda_- y)^n\right] \\ y_n &= \tfrac{1}{2\sqrt{(s^2/4)+t}}\left[(x+\lambda_+ y)^n - (x+\lambda_- y)^n\right] \end{aligned}\right\} \tag{30}$$

where $\lambda_{\pm} = \dfrac{s}{2} \pm \sqrt{\dfrac{s^2}{4} + t}$.
As a consequence, the Brahmagupta polynomials in two parameters satisfy the following three -term recurrences:

$$\left.\begin{aligned} x_{n+1} &= (2x+sy)x_n - \left(x^2+sxy-ty^2\right)x_{n-1}, x_0 = 1\ ,\ x_1 = x \\ y_{n+1} &= (2x+sy)y_n - \left(x^2+sxy-ty^2\right)y_{n-1}, y_0 = 0, y_1 = y \end{aligned}\right\}. \tag{31}$$

The first few Brahmagupta polynomials in two parameters are:

$$x_0 = 1,\ x_1 = x,\ x_2 = x^2 + ty^2,\ x_3 = x^3 + 3txy^2 + sty^3,$$

$$x_4 = x^4 + 4stx^3y + 6tx^2y^2 + stxy^3 + (t+s^2)y^4, \cdots ;$$

$$y_0 = 0,\ y_1 = y,\ y_2 = 2xy + sy^2,\ y_3 = 3x^2y + 3sxy^2 + (t+s^2)y^3,$$

$$y_4 = 4x^3y + 6sx^2y^2 + 4(t+s^2)xy^3 + s(2t+s^2)y^4, \cdots .$$

In [4], as a consequence of Theorem 1. it is shown that Brahmagupta polynomials are polynomial solutions of $t -$ *Cauchy's - Reimann* equations:

$$\left.\begin{aligned} \frac{\partial x_n}{\partial x} &= \frac{\partial y_n}{\partial y} = n\ x_{n-1} \\ \frac{\partial x_n}{\partial y} &= t\ \frac{\partial y_n}{\partial y} = n\ t\ y_{n-1} \end{aligned}\right\}. \tag{32}$$

As a further consequence, x_n and y_n are shown to satisfy the wave equation:

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{t}\frac{\partial^2}{\partial y^2}\right) U = 0. \tag{33}$$

The corresponding extended result is the following theorem :

Theorem 3 *The polynomials* $x_n(x, y, s, t)$ *and* $y_n(x, y, s, t)$ *satisfy the following second order linear partial differential equations :*

$$\left(\frac{\partial^2}{\partial x^2} + \frac{s}{t}\frac{\partial^2}{\partial x \partial y} - \frac{1}{t}\frac{\partial^2}{\partial y^2}\right) U = 0. \tag{34}$$

Proof Partial differentiation of (30) yields,

$$\frac{\partial}{\partial x} \ (x_n + \frac{s}{2}y_n) = \left(-\frac{s}{2}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) \ y_n = n \left(x_{n-1} + \frac{s}{2}y_{n-1}\right) \tag{35}$$

$$\frac{\partial}{\partial y}\left(x_n + \frac{s}{2}y_n\right) = n\left[\frac{s}{2}\left(x_{n-1} + \frac{s}{2}y_{n-1}\right) + \left(\frac{s^2}{4} + t\right) y_{n-1}\right] \tag{36}$$

$$\frac{\partial y_n}{\partial x} = n y_{n-1} \tag{37}$$

So we may simplify the above as follows-

$$\frac{\partial x_n}{\partial x} = -\left(s\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) y_n \tag{38}$$

$$\frac{\partial x_n}{\partial y} = -\frac{s}{2}\frac{\partial y_n}{\partial y} + \frac{s}{2}\left(-\frac{s}{2}\frac{\partial y_n}{\partial x} + \frac{\partial y_n}{\partial y}\right) + \left(\frac{s^2}{4} + t\right)\frac{\partial y_n}{\partial x} = t\frac{\partial y_n}{\partial x}$$

They naturally lead to

$$t\frac{\partial^2 y_n}{\partial x^2} + \frac{\partial}{\partial y}\left(s\ \frac{\partial}{\partial x} - \ \frac{\partial}{\partial y}\right) y_n = 0 \tag{39}$$

which is same as

$$\left(\frac{\partial^2}{\partial x^2} + \frac{s}{t}\ \frac{\partial^2}{\partial x\,\partial y} - \frac{1}{t}\frac{\partial^2}{\partial y^2}\right)\ y_n = 0 \tag{40}$$

Also, the Partial differential equation for x_n may be derived as follows-

$$\frac{\partial x_n}{\partial x} + \frac{s}{t}\frac{\partial x_n}{\partial y} = \frac{\partial y_n}{\partial y} \tag{41}$$

$$\frac{1}{t}\frac{\partial x_n}{\partial y} = \frac{\partial y_n}{\partial x} \tag{42}$$

As a direct consequence, x_n satisfies the following Partial differential equation-

$$\left(\frac{\partial^2}{\partial x^2} + \frac{s}{t}\ \frac{\partial^2}{\partial x\,\partial y} - \frac{1}{t}\frac{\partial^2}{\partial y^2}\right)\ x_n = 0 \tag{43}$$

□

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Biharmonic Slant Helices According to Bishop Frame in $\mathbb{E}^3$

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Abstract: In this paper, we study biharmonic slant helices in $\mathbb{E}^3$. We give some characterizations for biharmonic slant helices with Bishop frame in $\mathbb{E}^3$.

Key Words: Slant helix, biharmonic curve, bishop frame.

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§1. Introduction

In 1964, J. Eells and J.H. Sampson introduced the notion of poly-harmonic maps as a natural generalization of harmonic maps [1].

Firstly, harmonic maps $f:(M,g)\longrightarrow(N,h)$ between Riemannian manifolds are the critical points of the energy

$$E(f)=\frac{1}{2}\int_M |df|^2 v_g, \tag{1.1}$$

and they are therefore the solutions of the corresponding Euler–Lagrange equation. This equation is given by the vanishing of the tension field

$$\tau(f)=\operatorname{trace}\nabla df. \tag{1.2}$$

Secondly, as suggested by Eells and Sampson in [1], we can define the bienergy of a map f by

$$E_2(f)=\frac{1}{2}\int_M |\tau(f)|^2 v_g, \tag{1.3}$$

and say that is biharmonic if it is a critical point of the bienergy.

Jiang derived the first and the second variation formula for the bienergy in [3], showing that the Euler–Lagrange equation associated to E_2 is

$$\tau_2(f)=-\mathcal{J}^f(\tau(f))=-\Delta\tau(f)-\operatorname{trace}R^N(df,\tau(f))\,df=0 \tag{1.4}$$

where $\mathcal{J}^f$ is the Jacobi operator of f. The equation $\tau_2(f)=0$ is called the biharmonic equation. Since $\mathcal{J}^f$ is linear, any harmonic map is biharmonic. Therefore, we are interested in proper biharmonic maps, that is non-harmonic biharmonic maps.

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In this paper, we study biharmonic slant helices in $\mathbb{E}^3$. We give some characterizations for biharmonic slant helices with Bishop frame in $\mathbb{E}^3$.

§2 Preliminaries

To meet the requirements in the next sections, here, the basic elements of the theory of curves in the space $\mathbb{E}^3$ are briefly presented.

The Euclidean 3-space $\mathbb{E}^3$ provided with the standard flat metric given by

$$\langle\ ,\ \rangle = dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbb{E}^3$. Recall that, the norm of an arbitrary vector $a \in \mathbb{E}^3$ is given by $\|a\| = \sqrt{\langle a, a\rangle}$. γ is called a unit speed curve if velocity vector v of γ satisfies $\|a\| = 1$.

Denote by $\{T, N, B\}$ the moving Frenet–Serret frame along the curve γ in the space $\mathbb{E}^3$. For an arbitrary curve γ with first and second curvature, κ and τ in the space $\mathbb{E}^3$, the following Frenet-Serret formulae is given

$$\begin{aligned}
\mathbf{T}' &= \kappa\mathbf{N}, \\
\mathbf{N}' &= -\kappa\mathbf{T} + \tau\mathbf{B}, \\
\mathbf{B}' &= -\tau\mathbf{N},
\end{aligned}$$

where

$$\begin{aligned}
\langle\mathbf{T}, \mathbf{T}\rangle &= \langle\mathbf{N}, \mathbf{N}\rangle = \langle\mathbf{B}, \mathbf{B}\rangle = 1, \\
\langle\mathbf{T}, \mathbf{N}\rangle &= \langle\mathbf{T}, \mathbf{B}\rangle = \langle\mathbf{N}, \mathbf{B}\rangle = 0.
\end{aligned}$$

Here, curvature functions are defined by $\kappa = \kappa(s) = \|\mathbf{T}(s)\|$ and $\tau(s) = -\left\langle \mathbf{N}, \mathbf{B}'\right\rangle$.

Torsion of the curve γ is given by the aid of the mixed product

$$\tau(s) = \frac{[\gamma', \gamma'', \gamma''']}{\kappa^2}.$$

In the rest of the paper, we suppose everywhere $\kappa(s) \neq 0$ and $\tau(s) \neq 0$.

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. One can express parallel transport of an orthonormal frame along a curve simply by parallel transporting each component of the frame. The tangent vector and any convenient arbitrary basis for the remainder of the frame are used. The Bishop frame is expressed as

$$\mathbf{T}' = k_1\mathbf{M}_1 + k_2\mathbf{M}_2, \mathbf{M}_1' = -k_1\mathbf{T}, \mathbf{M}_2' = -k_2\mathbf{T}. \tag{2.1}$$

Here, we shall call the set $\{\mathbf{T}, \mathbf{M}_1, \mathbf{M}_1\}$ as Bishop trihedra and k_1 and k_2 as Bishop curvatures. The relation matrix may be expressed as

$$\begin{aligned}
&\mathbf{T} = \mathbf{T}, \\
&\mathbf{N} = \cos\theta\,(s)\,\mathbf{M}_1 + \sin\theta\,(s)\,\mathbf{M}_2, \\
&\mathbf{B} = -\sin\theta\,(s)\,\mathbf{M}_1 + \cos\theta\,(s)\,\mathbf{M}_2,
\end{aligned}$$

where $\theta(s) = \arctan \frac{k_2}{k_1}$, $\tau(s) = \theta'(s)$ and $\kappa(s) = \sqrt{k_1^2 + k_2^2}$. Here, Bishop curvatures are defined by

$$k_1 = \kappa(s) \cos \theta(s), \qquad k_2 = \kappa(s) \sin \theta(s).$$

On the other hand, we get

$$\begin{aligned} \mathbf{T} &= \mathbf{T}, \\ \mathbf{M}_1 &= \cos \theta(s) \mathbf{N} - \sin \theta(s) \mathbf{B}, \\ \mathbf{M}_2 &= \sin \theta(s) \mathbf{N} + \cos \theta(s) \mathbf{B}. \end{aligned}$$

§3. Biharmonic curves in $\mathbb{E}^3$

Biharmonic equation for the curve γ reduces to

$$\nabla_{\mathbf{T}}^3 \mathbf{T} - R(\mathbf{T}, \nabla_{\mathbf{T}} \mathbf{T}) \mathbf{T} = 0, \tag{3.1}$$

that is, γ is called a biharmonic curve if it is a solution of the equation (3.1).

Theorem 3.1 *$\gamma : I \longrightarrow \mathbb{E}^3$ is a unit speed biharmonic curve if and only if*

$$\begin{aligned} k_1^2 + k_2^2 &= C, \\ k_1'' - k_1^3 - k_1 k_2^2 &= 0, \\ k_2'' - k_2^3 - k_2 k_1^2 &= 0, \end{aligned} \tag{3.2}$$

where C is non-zero constant of integration.

Proof Using the bishop equations (2.1) and biharmonic equation (3.1), we obtain

$$(-3k_1' k_1 - 3k_2' k_2)\mathbf{T} + (k_1'' - k_1^3 - k_1 k_2^2)\mathbf{M}_1 + (k_2'' - k_2^3 - k_2 k_1^2)\mathbf{M}_2 - R(\mathbf{T}, \nabla_{\mathbf{T}} \mathbf{T}) \mathbf{T} = \mathbf{0}. \tag{3.3}$$

In $\mathbb{E}^3$, the Riemannian curvature is zero, we have

$$(-3k_1' k_1 - 3k_2' k_2)\mathbf{T} + (k_1'' - k_1^3 - k_1 k_2^2)\mathbf{M}_1 + (k_2'' - k_2^3 - k_2 k_1^2)\mathbf{M}_2 = 0. \tag{3.4}$$

By (3.4), we see that γ is a unit speed biharmonic curve if and only if

$$\begin{aligned} -3k_1' k_1 - 3k_2' k_2 &= 0, \\ k_1'' - k_1^3 - k_1 k_2^2 &= 0, \\ k_2'' - k_2^3 - k_2 k_1^2 &= 0. \end{aligned} \tag{3.5}$$

These, together with (3.5), complete the proof of the theorem. □

Corollary 3.2 *$\gamma : I \longrightarrow \mathbb{E}^3$ is a unit speed biharmonic curve if and only if*

$$\begin{aligned} k_1^2 + k_2^2 &= C \neq 0, \\ k_1'' - C k_1 &= 0, \\ k_2'' - C k_2 &= 0, \end{aligned} \tag{3.6}$$

where C *is constant of integration.*

Theorem 3.3 *Let* $\gamma : I \longrightarrow \mathbb{E}^3$ *is a unit speed biharmonic curve, then*

$$\begin{aligned} k_1^2(s) + k_2^2(s) &= C, \\ k_1(s) &= c_1 e^{\sqrt{C}s} + c_2 e^{-\sqrt{C}s}, \\ k_2(s) &= c_3 e^{\sqrt{C}s} + c_4 e^{-\sqrt{C}s}, \end{aligned} \tag{3.7}$$

where C, c_1, c_2, c_3, c_4 *are constants of integration.*

Proof Using (3.6), we have (3.7). □

Corollary 3.4 *If* $c_1 = c_3$ *and* $c_2 = c_4$, *then*

$$k_1(s) = k_2(s). \tag{3.8}$$

Definition 3.5 *A regular curve* $\gamma : I \longrightarrow \mathbb{E}^3$ *is called a slant helix provided the unit vector* $\mathbf{M}_1$ *of the curve* γ *has constant angle* θ *with some fixed unit vector* u, *that is*

$$g(\mathbf{M}_1(s), u) = \cos\theta \text{ for all } s \in I. \tag{3.9}$$

The condition is not altered by reparametrization, so without loss of generality we may assume that slant helices have unit speed. The slant helices can be identified by a simple condition on natural curvatures.

Theorem 3.6 *Let* $\gamma : I \longrightarrow \mathbb{E}^3$ *be a unit speed curve with non-zero natural curvatures. Then,* γ *is a slant helix if and only if*

$$\frac{k_1}{k_2} = \text{constant}. \tag{3.10}$$

Proof Differentiating (3.9) and by using the Bishop frame (2.1), we find

$$g(\nabla_{\mathbf{T}}\mathbf{M}_1, u) = g(k_1\mathbf{T}, u) = k_1 g(\mathbf{T}, u) = 0. \tag{3.11}$$

From (3.9), we get

$$g(\mathbf{T}, u) = 0.$$

Again differentiating from the last equality, we obtain

$$\begin{aligned} g(\nabla_{\mathbf{T}}\mathbf{T}, u) &= g(k_1\mathbf{M}_1 + k_2\mathbf{M}_2, u) \\ &= k_1 g(\mathbf{M}_1, u) + k_2 g(\mathbf{M}_2, u) \\ &= k_1\cos\theta + k_2\sin\theta = 0. \end{aligned}$$

Using above equation, we get

$$\frac{k_1}{k_2} = -\tan\theta = \text{constant}.$$

The converse statement is trivial. This completes the proof. □

Theorem 3.7 . *Let $\gamma : I \longrightarrow \mathbb{E}^3$ be a unit speed biharmonic slant helix with non-zero natural curvatures. Then,*

$$k_1 = \text{constant and } k_2 = \text{constant}. \tag{3.12}$$

Proof Suppose that γ be a unit speed biharmonic slant helix. From (3.10) we have

$$k_1 = \sigma k_2. \tag{3.13}$$

where σ is a constant.

On the other hand, using first equation of (3.6), we obtain that k_2 is a constant. Similarly, k_1 is a constant.

Hence, the proof is completed. □

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Combinatorial Optimization in VLSI Hypergraph Partitioning using Taguchi Methods

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Abstract: This work addresses the methods to solve Very Large Scale Integration (VLSI) circuit partitioning problem with dual objectives, viz., 1. Minimizing the number of interconnection between partitions, that is, the cut size of the circuit and 2. Balancing the area occupied by the partitions. In this work an efficient hybrid Genetic Algorithm (GA) incorporating the Taguchi method as a local search mechanism has been developed to solve both bipartitioning and recursive partitioning problems in VLSI design process. The systematic reasoning ability of the Taguchi method incorporated after the crossover operation of GA, has improved the searching ability of GA. The proposed Hybrid Taguchi Genetic Algorithm (HTGA) has been tested with fifteen popular bench mark circuits of ISCAS 89 (International Symposium on Circuit and Systems-89). The results of experiments conducted, have proved that HTGA is able to converge faster in reaching the nearer-to-optimal solutions. The performance of the proposed HTGA is compared with that of the standard GA and Tabu Search method reported in the literature. It is found that the proposed HTGA is superior and consistent both in terms of number of iterations required to reach nearer-to-optimal solution and also the solution quality.

Key Words: VLSI, partitioning, genetic algorithm, Taguchi method, cut size, multi-partitioning.

AMS(2000): 49J35

§1. Introduction

During the Very Large Scale Integration (VLSI) design process, the complex circuit comprising of elements like gates, buffers, Input/Output ports which are inter connected by wires is divided into subsets, that is, modules [10,16] as the first step. This partitioning of the circuit into smaller modules is essential to reduce the problem complexity of the VLSI physical design

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problem. Proper partitioning of a VLSI circuit will result in minimum total area occupied by all the elements of the circuit, and reduction in the total length of interconnecting wires between the elements, which will in turn minimize the power dissipation and time delay during its operation. To achieve these objectives of VLSI design problem, the complex VLSI circuit should be partitioned into smaller sub modules such that the number of wires passing between the elements of different modules is kept minimum. For a particular partition, the sum total of number of wires passing between the modules is known as cutsize of the partition. A partition with modules occupying equal area will largely help in the later part of the VLSI design process namely floorplanning, placement and routing. Hence, partitioning of VLSI circuit should be done in such a way that, all the modules occupy more or less equal area or in other words the uneven distribution of area among the modules, that is, imbalance in area should be kept minimum. Hence in this work, both these objectives (i) minimizing the cutsize and (ii) minimizing the area imbalance among the modules are considered for solving the VLSI partitioning problem.

VLSI circuit partitioning is proved to be an intractable problem [14] and only satisfactory solutions to the different problem instances are being generated by designing suitable metaheuristic algorithms. In this research work, an attempt is made to design a suitable metaheuristic algorithm capable of producing consistent solution with lesser number of iterations for a wider range of VLSI circuit problem.

§2. Literature survey

B.W.Kernighan and S.Lin proposed the group migration algorithm (KL algorithm) [12] for graph partitioning problem which through the years of use has been proved to be very efficient. However KL algorithm is designed only for bipartitioning the given circuit. C.M.Fiduccia and R.M.Mattheyses (FM) improved the KL algorithm by introducing an elegant bucket sorting technique [7]. However, FM algorithm was able to provide satisfactory solutions only for smaller to medium size problems and also only for bipartitioning the circuit. Later Cong.J (1994) developed k-way net based multi way partitioning algorithm to produce better quality solutions than the FM algorithm but only for smaller size problems. Mean time hMetis [24] and other Multilevel Clustering algorithms (MLC) were developed [8] based on the flat partitioning methodology with an aim of further minimizing the cutsize. Later, the Multilevel Partitioning algorithm (MLP) that is also based on the flat partitioning methodology, was developed by Jong-Sheng (2003) and its performance surpassed the result produced by hMetis and MLC in terms of minimal cutsize. However it is proved that flat multiway partitioning approach could produce better quality results for smaller size integrated circuits [17,18], and due to the space complexity ($O(N.K\ (K\text{-}1)$) where N denotes the number of cells) and poor flexibility, the approach is less efficient with larger size integrated circuits. The method of recursive partitioning evolved by Aeribi.S [3] is found to be performing better than the flat partitioning methodology interms of solution quality but at the cost of additional computational load. Sadiq.M.Sait developed metaheuristic algorithms [16] based on Genetic Algorithm (GA) and Tabu search (TS) to address relatively larger size problems and with multiple objectives. In his work he has

proved that though GA is able to produce quality solutions for smaller size circuits and Tabu search outperforms GA in terms of both quality of the solution and execution time even for the larger circuits.

In this work, with an emphasis on solution quality, research focus is retained to improve upon the recursive partitioning methodology, inspite of its heavy computational requirement compared to the flat partitioning methodology. Also to address the problem complexity of VLSI multi partitioning problem, which is NP-hard, an attempt is made to develop a metaheuristic algorithm based on the robust and versatile tool, GA. To overcome the inherent scalability issue with the GA, the Taguchi method, a robust design approach is incorporated in the genetic search process.

§3. Problem formulation

Any VLSI circuit consisting of more than one component or element (that is either a gate or flip flop or buffer) can be represented in the form of a hyper graph $H(V,E)$. $V = \{v_1, v_2, v_3 \cdots v_n\}$ is the set of nodes representing the elements used in the circuit and $E = \{e_1, e_2, e_3 \cdots e_n\}$ is the set of edges representing all the required connections between the elements. The aim of the work is to split the given hyper graph into required number of partitions with minimum number of inter connections between the partitions (namely the cutsize) and also with minimal area imbalance between the modules, that is, the uneven distribution of area among the partitions. An attempt to minimize the number of interconnecting wires between two modules by placing the elements associated in the interconnectivity, together in one module will result in increase in area imbalance between the two modules, and vice versa. Hence in order to achieve the above said two contradicting objectives concurrently, the following combined objective function is constructed.

The Combined Objective Function (COF):

$$COF = Minimize\ [(\alpha_1 * F_1) + (\alpha_2 * F_2)] \tag{1}$$

where,
F_1 = Cutsize (given in (2))
F_2 = Area imbalance between the circuits (given in (3))
α_1 = Weightage factor assigned to the cutsize
α_2 = Weightage factor assigned to the area imbalance
The function [23] for cutsize (F_1) is:

$$F_1 = \sum_{\forall r \in E} \left(\sum_{i=1}^{(|Q_r|-1)} (-1)^{i+1} c_i^{Q_r} - 2F \prod_{j=1}^{|Q_r|} x_j \right) \tag{2}$$

where,
Q_r= Set of assignment variables for all non Input/Output components on net (edges) r

$$F = \begin{cases} 1\ if\ |Q_r|\ is\ even \\ 0\ otherwise \end{cases}$$

E = Set of edges

${C_i}^{Q_r}$ = Combinations of the set Q_r taken i at a time

x_j = Set of nodes

The function for area imbalance (F_2) is:

$$F_2 = \beta_1 - \beta_2 \tag{3}$$

where,

$\beta_1 = max \{ |P| : P \text{ is a partition } \}$

$\beta_2 = min \{ |P| : P \text{ is a partition } \}$

$|P|$ = Number of elements in a partition

§4. Proposed methodology

A GA based heuristic namely Hybrid Taguchi Genetic Algorithm (HTGA) is proposed in this work, to solve the VLSI circuit partitioning problem with dual objectives of minimizing the cutsize and minimizing the area imbalance among the partitions. The proposed algorithm is tested with fifteen popular bench mark circuits of ISCAS89, and its performance is compared with that of the other metaheuristics reported in the literature.

4.1 Genetic Algorithm

Genetic algorithm operates on the principle of *survival-of-the-fittest*, where weak individuals die, while stronger ones survive and bear many offspring and breed children, which often inherit qualities that are, in many cases superior to their parent's qualities [14]. GA begins with a population offspring (individuals- representing the design/decision variables) created randomly. Thereafter, each string in the population is evaluated to find its fitness value (that is, the objective function value of the given optimization problem). The operators *Selection, Crossover and Mutation* are used to create a new and better population. The new population is further evaluated for the fitness values and tested for termination. If the termination criteria are not met, the population is interactively operated by the above genetic operators and evaluated. One cycle of these genetic operations and the evaluation procedure is known as a *generation* in GA terminology. The generation cycle is continued until the termination criterion is met.

4.2 Taguchi Method

Taguchi method is a robust design approach, which uses many ideas from statistical experimental design for evaluating and implementing improvements in products, processes and equipment [21,9]. The fundamental principle of Taguchi method is to improve the quality of a product by minimizing the effect of the causes of variation without eliminating the inevitable causes.

The two major tools used in the Taguchi method are:

1. *Orthogonal arrays (OA) which are used to study many design parameters simultaneously,*
2. *Signal-to-Noise Ratio (SNR) which measures quality.*

For instance, let there be an optimization problem whose solution is influenced by, say seven factors and each of these factors can be at any of the two levels. If the objective is to find a suitable level for each factor to find an optimal solution, then the total number of possible experiments is 2^7 to find the optimal solution. An orthogonal array (OA), an example shown in Table 1, represents a set of recommended limited number of experiments, (eight for the example shown in Table 1, needed to find a suitable level for each factor to achieve an optimal solution at a faster rate. Thus, with the help of only these 8 experiments out of a total 2^7 possible experiments, the best solution can be found with each factor being at a suitable level. The orthogonal arrays are represented as $L_n(x^{n-1})$, where $n = 2^k$ is the number of experimental runs, k is a positive integer, x is the number of levels for each factor and $n-1$ is the number of columns in an orthogonal array. The example OA is shown in the Table 1, is of $L_8(2^7)$ type.

The second tool of Taguchi method, the SNR, is used to find which level is suitable for each factor; SNR calculation is discussed with an example in Section **??**. In communication engineering parlance, the Signal to Noise Ratio means the measure of signal quality, which corresponds to the solution quality in Taguchi method. While conducting each experiment as per the orthogonal array, the objective function value is computed, and the effect of each of the two levels on each factor in contributing to the objective function value is computed. A level to a particular factor, which gives the maximum effect in contribution to the objective function value, is optimal for the concerned factor. As the effect is maximum for this level, it is said to have maximum influence or the maximum *Signal to Noise Ratio* (SNR) and so considered as optimal level for the factor. With the conduct of all the experiments as per the orthogonal array, the solution obtained with optimal level for each factor, is the optimum solution for the given optimization problem.

4.3 Hybrid Taguchi Genetic Algorithm (HTGA)

In the proposed Hybrid Taguchi Genetic Algorithm (HTGA) to solve the VLSI partitioning problem, the Taguchi method is embedded within GA, between the crossover and mutation operations, to improve all the solutions of the intermediate population obtained after the crossover operation and before subjected to the subsequent mutation operation.

The proposed HTGA is designed to generate multi-partitioning solutions for larger size VLSI problems through the recursive approach, recomented by Areibi.S [3]. The adapted recursive approach applies bipartitioning recursively until the desired number of partition is obtained, which is illustrated in the example shown in Fig. 1, where a single VLSI circuit is recursively partitioned into eight partitions.

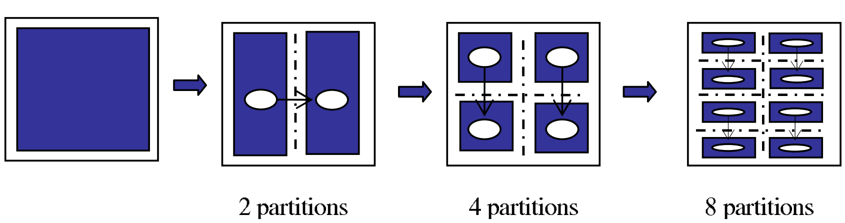

Figure 1: Recursive partitioning of a VLSI circuit

In HTGA, genotype representation is used to code a feasible solution as a chromosome [4,14]. The zeros and ones in a chromosome represents either of the two partitions they belong to. In case of multiple partitions through recursive partitioning, each of the divided chromosomes representing each partition will have zeros and ones representing either of the two sub partitions.

A bipartition solution of a VLSI circuit having components v_1, v_2, v_3, v_4, v_5 and v_6 shown in the Fig. 2 is encoded as a solution chromosome as shown in Fig. 3. The digit one represents that the element is present in the partition P_1 otherwise in P_2.

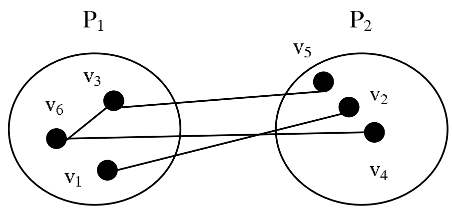

Figure 2: A bipartitioning solution of the example VLSI circuit

v_1	v_2	v_3	v_4	v_5	v_6
1	0	1	0	0	1

Figure 3: Chromosome representation of bipartition solution

When the bipartition solution shown in Fig.3 is further partitioned through recursive method, that is, when P_1 is partitioned into $P_{1(a)}$ and $P_{1(b)}$ and P_2 is partitioned into $P_{2(a)}$ and $P_{2(b)}$, a sample solution shown in Fig.4 is encoded as a solution chromosome as shown in Fig.5.

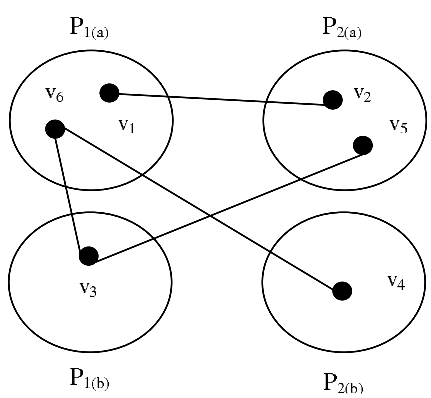

Figure 4: A recursive partitioning solution of the example VLSI circuit

In the proposed HTGA, the random initial population of partitioning solutions is subjected to selection and crossover operations. The resultant intermediate population obtained through the cross over operations is fed to the local search mechanism, Taguchi method module of the

v_1	v_3	v_6	v_2	v_4	v_5
1	0	1	1	0	1

Figure 5: Chromosome representation of the solution with four partitions

HTGA. This phase of the HTGA creates a new improved intermediate population of same size with each solution entirely different from the initial solutions of the intermediate population resulted out of crossover operation of GA.

The algorithm shows the Taguchi phase in HTGA.

Algorithm

Encode the random initial population of solution
Do while the termination criteria is not met
  Step 1: Perform Reproduction
  Step 2: Perform Crossover
  Step 3: Taguchi Method
    a: Select a suitable orthogonal array
    Do while the size of the population is reached
      Do while an improved solution is found
        Step b: Random selection of pair of chromosome.
        Step c: Calculate SNRs.
          Compute Effect of Factors.
          Select the optimal bit
        Step d: Construct new chromosome
      End Do
    End Do
  Step 4: Perform Mutation
End Do
Decode the best solution in the final population to get the optimal partition.

In each iteration of this phase, a pair of chromosomes, say X and Y are selected at random from the intermediate population and a better chromosome Z is evolved by choosing each gene either from chromosome X (level 1) or from chromosome Y (level 2). The Taguchi method of producing a better chromosome Z from a randomly chosen two chromosomes X and Y is illustrated in Table 2. Selection of suitable level is done by conducting eight experiments as per the example orthogonal array, shown in Table 1. For each experiment the functional value which is *COF* of experimental chromosome is computed. As the problem is minimization problem, the signal to noise ratio, SNR (η_i) for each experiment i is computed as a reciprocal of *COF* value of the experimental chromosome. Having calculated the SNR value for all the experiments, for each gene, the effect of choosing from level 1 (chromosome X) or level 2 (chromosome Y) chromosome is computed as equations 4 and 5.

$$Ef_1 = \sum_{i=1}^{n} SNR(\eta_i),\ when\ gene\ i\ is\ belongs\ to\ level\ 1 \tag{4}$$

$$Ef_2 = \sum_{i=1}^{n} SNR(\eta_i),\ when\ gene\ i\ is\ belongs\ to\ level\ 2 \tag{5}$$

The gene is selected from the level for which the effect of factor Ef_i is maximum and the improved chromosome Z is thus constructed with all such selected genes in their respective positions.

The above said iteration is repeated by selecting another pair of chromosomes from the intermediate population and a new chromosome is created. The procedure is repeated till the new intermediate population of required size is created. This improved intermediate population is fed to the subsequent mutation operator of generation cycle of GA. The generation cycle of HTGA is repeated till the termination criterion is met.

§5. Results and discussions

The proposed algorithm, HTGA was coded in C++ and experiments were conducted in an IBM Pentium D PC with 3.20 GHz Processor. The HTGA was tested with fifteen number of ISCAS89 (International Symposium of Circuit And Systems) benchmark circuits. The details of the benchmarks are shown in Table 3. To measure the effect of Taguchi method in the proposed HTGA, the performance of HTGA is compared with that of the standard template of GA, that is, a genetic algorithm without the hybridization of Taguchi method. To make the comparison on a common platform the standard GA is also coded in C++, run on the same machine and tested with the same benchmark circuits.

In the proposed HTGA tournament selection is used for reproduction operation, Single cut point crossover is used in the crossover operation and Flap bit mutation is used for mutation operation. The parameters used in HTGA are as below.

1. Population Size = 20
2. Crossover probability (P_c) = 0.6
3. Mutation probability (P_m) = 0.01
4. Termination Criterion = A predefined number of iterations for a given circuit or a predefined satisfactory COF value, whichever occurs first.
5. Orthogonal array used in the Taguchi experimentation is $L_8(2^7)$.

The best values for the individual parameters are fixed by conducting trials and on satisfactory performance. The crossover probability P_c was varied from 0.4 to 0.9, and the GA is found able to converge faster with a crossover probability P_c of value 0.6. Similarly the mutation probability P_m was varied between 0.001 to 0.1 and the GA with the mutation probability P_m of value 0.01 is found able to retain more number of better solution than worse solution at the end of GA cycle.

For all the bench mark circuits taken in this work, the proposed algorithm HTGA is able to outperform the standard Genetic Algorithm both in bipartitioning application and so in recursive partitioning application, again both in terms of number of iterations required to reach a nearer-to-optimal solution and also in terms of the quality of the solution, that is the absolute value of COF. The results of this comparative study between GA and HTGA in bipartitioning

and in recursive partitioning (four partitions) are shown in Tables 4 and 5 respectively.

It can be seen from both Tables 4 and 5, that the CPU time taken by HTGA is higher compared to the standard GA for smaller circuit, which may be attributed to the additional computational load required because of the Taguchi method of HTGA. However it can be also seen from these tables that, for larger circuits, the CPU time taken by HTGA is substantially lower than standard GA, which can be attributed to the efficiency of HTGA in reaching the solutions with lesser number of generation cycles.

It is observed that because of the Taguchi method after the crossover operation, HTGA is able to converge at a faster rate than that of the standard GA, which is explained with a sample benchmark problem S832 in Fig.6.

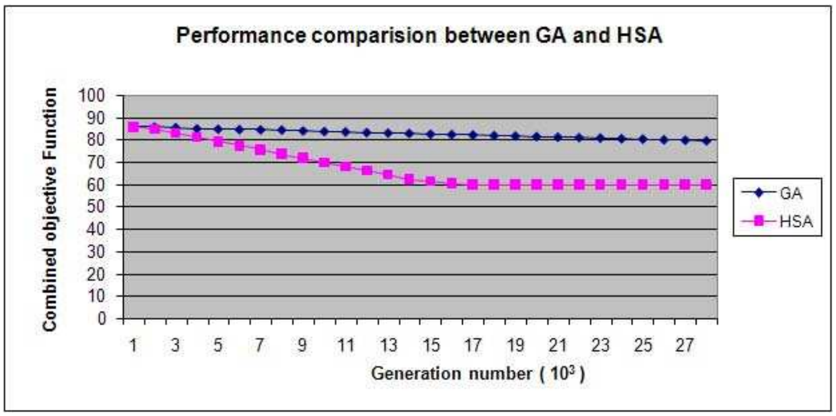

Figure 6: Convergence comparison between GA and HTGA for the benchmark problem S832

For each of the fifteen ISCAS89 benchmark circuits the experiment is conducted with 25 sets of different initial random populations, again with each initial random population the experiment is repeated 100 times to access the consistency rate of the solution produced by the proposed HTGA. The percentage consistency rate is computed as {(number of trials getting COF value within five percent of the best found COF value /total number of trials)*100}. The summary of the findings are shown in Table 6, which exhibit that the consistency rate of proposed HTGA is considerably higher than the normal GA.

The performance of the HTGA is also compared with that of two meta heuristics, reported in the literature [16] viz (i). GA based heuristic, (ii). Tabu Search based heuristic. The cutsize obtained by these heuristic and the proposed HTGA is shown in Table 7.

It can be seen from the Table 7, that though the GA based heuristic proposed in the literature [16] is effective in minimizing the cutsize for smaller benchmark circuits, the Tabu Search based heuristic given in the literature is able to outperform the GA for larger benchmark circuits. The proposed HTGA overcomes this issue and produces lesser cutsize for all the benchmark circuits except S386 and S5378. For these two circuits cutsize produced by HTGA is marginally higher than the Tabu Search based meta heuristics but lower than GA based heuristics. The effectiveness of HTGA in producing better quality solutions could be attributed to the systematic reasoning ability of the Taguchi method, which is built in the proposed HTGA.

Again the proposed HTGA may be made to surpass the performance of TS for the circuits S386 and S5378 by designing an improved OA even with more than 2 levels, (if required), which is a part of the scope for future work.

As the hMetis [24] algorithm, and other algorithms such as MLP, MLC mentioned in the literature in section 2 are suited for only flat partitioning [3] and are capable of producing solutions even for very large size problems with appreciably lesser time with the objective of producing solution with satisfactory quality level, the run time of hMetis, MLP, MLC cannot be compared with that of the proposed HTGA, which uses recursive partitioning methodology and whose solution quality is expected to be much higher than that of the flat partitioning methodology [3,17-18].

Due to the recursive nature and a larger number of computations involved in OA, HTGA needs more computational time for larger scale benchmarks. However this issue could be addressed by constructing dedicated OA with more number of factors. And grouping of higher cardinality edges in a particular partition (P_i) instead of doing random initial population generation, which is again the scope for future work.

§6. Conclusion

In this work, an attempt is made to solve the VLSI circuit partitioning problem with an objective of minimizing the cutsize, that is, the number of wires passing between the partitions and also balancing the area between the partitions. An efficient hybrid Genetic Algorithm incorporating Taguchi method as a local search mechanism, named as, Hybrid Taguchi Genetic Algorithm (HTGA) has been developed to solve both the bipartitioning and recursive partitioning problem in the VLSI design process. The proposed HTGA is tested with a wide range of ISCAS89 benchmark circuits and its performance is compared with that of a standard GA (without the use of Taguchi as a local search tool) and it is found that HTGA out performs the standard GA both in terms of solution quality and the number of iterations required for reaching the nearer-to-optimal solution, due to the systematic reasoning ability of the Taguchi method. The experimentation with proposed HTGA was also repeated with the same and different input data sets and it was found that the proposed HTGA is consistent in producing quality solutions. The performance of HTGA is also compared with that of the GA and Tabu Search based meta heuristics reported in the literature. And it is found that the proposed HTGA is able to give better solutions than the GA based heuristics for all the benchmark circuits considered in this work. Compared to the Tabu Search based heuristic, the proposed HTGA is able to produce better solution for all the benchmark circuits except S386 and S5378. Again HTGA may be made to surpass the performance of TS for the circuits S386 and S5378 by designing an improved orthogonal array (OA) even with more than 2 levels (if required) which is a part of the scope for the future work.

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Appendix:

Table 1: An example Orthogonal Array, $L_8(2^7)$

	Factors						
	1	2	3	4	5	6	7
Experiment	A	B	C	D	E	F	G
number	Levels assigned						
1	1	1	1	1	1	1	1
2	1	1	1	2	2	2	2
3	1	2	2	1	1	2	2
4	1	2	2	2	2	1	1
5	2	1	2	1	2	1	2
6	2	1	2	2	1	2	1
7	2	2	1	1	2	2	1
8	2	2	1	2	1	1	2

Table 2: An example calculation of Taguchi method.

Step a: Select a suitable two level orthogonal array, say $L_8(2^7)$ shown in Table 1

Step b: Randomly select two chromosomes from the intermediate crossover population

Chromosome X : 1 0 1 1 1 1 1 (level 1)
Chromosome Y : 0 1 1 1 0 1 0 (level 2)

Step c: Taguchi Experiment

	Factors								
	1	2	3	4	5	6	7		
Experiment	A	B	C	D	E	F	G	Function value COF_i	$SNR(\eta_i)$
1	1	0	1	1	1	1	1	3.5	0.28
2	1	0	1	1	0	1	0	2.0	0.50
3	1	1	1	1	1	1	1	4.0	0.25
4	1	1	1	1	0	1	1	5.0	0.20
5	0	0	1	1	0	1	0	3.0	0.33
6	0	0	1	1	1	1	1	3.0	0.33
7	0	1	1	1	0	1	1	3.0	0.33
8	0	1	1	1	1	1	0	5.0	0.20
Ef_1	1.23	1.44	1.31	1.19	1.06	1.14	1.14		
Ef_2	1.19	0.98	1.10	1.23	1.36	1.41	1.28		
Optimal Level	1	1	1	2	2	2	2		

Step d: Construct a new chromosome

Optimal Chromosome Z	1	0	1	1	0	1	0

Table 3: Details of ISCAS89 benchmark problems tested with HTGA

S.NO	Benchmark Circuit Code	Number of Elements	Number of Interconnections
1	S27	18	13
2	S208	117	108
3	S298	136	130
4	S386	172	165
5	S641	433	410
6	S832	310	291
7	S953	440	417
8	S1196	561	547
9	S1238	540	526
10	S1488	667	648
11	S1494	661	642
12	S5378	2994	2944
13	S9234	5845	5822
14	S13207	8652	8530
15	S15850	10384	10296

Table 4: Performance comparison between GA and HTGA in bipartitioning

Benchmark Circuit	Standard Genetic Algorithm				
	Cut size (F_1)	Area (F_2)	COF	No. of Generations	CPU time (s)
S27	3	2	2.5	2	2
S208	30	20	25	25641	552
S298	15	26	20.5	4872	95
S832	40	84	62	28436	278
S386	38	101	69.5	7985	165
S641	47	128	87.5	33700	1506
S953	95	139	117	27741	600
S1196	110	13	61.5	6654	396
S1238	98	65	81.5	4385	380
S1488	104	10	57	9359	1058
S1494	104	18	61	8659	1102
S5378	541	30	285.5	12658	1956
S9234	1082	42	562	28958	4558
S13207	1602	80	841	30258	6582
S15850	2186	24	1105	38598	8965
HTGA					
S27	3	1	2	2	2
S208	27	18	22.5	9189	659
S298	13	25	19	2346	112
S832	39	74	56.5	18849	290
S386	32	95	63.5	3339	170
S641	44	117	80.5	29221	1600
S953	84	141	112.5	21080	556
S1196	102	13	57.5	4159	398
S1238	73	74	73.5	2958	302
S1488	92	18	55	8158	650
S1494	101	19	60	6858	520
S5378	463	36	249.5	9958	952
S9234	915	46	480.5	12554	2858
S13207	1328	91	709.5	20587	4965
S15850	1665	30	847.5	25987	4895

Table 5: Performance comparison between GA and HTGA in Multi-Partitioning(4-Partitions)

Benchmark Circuit	Standard Genetic Algorithm				
	Cut size (F_1)	Area (F_2)	COF	No. of Generations	CPU time (s)
S27	6	3	4.5	11	15
S208	45	19	32	37580	705
S298	55	19	37	10144	192
S832	97	27	62	48325	596
S386	72	105	88.5	16470	421
S641	99	83	91	49435	3254
S953	102	115	108.5	45434	1000
S1196	123	8	65.5	12065	821
S1238	118	49	83.5	8658	859
S1488	112	6	59	15285	3548
S1494	123	11	67	16258	2658
S5378	552	25	288.5	24585	4586
S9234	1125	33	579	45866	5486
S13207	1658	45	851.5	60258	8456
S15850	2103	18	1060.5	66558	12455
		HTGA			
S27	5	2	3.5	10	13
S208	34	20	27	17125	802
S298	48	22	35	4913	185
S832	85	21	53	26218	630
S386	69	98	83.5	15264	513
S641	80	52	66	34934	3951
S953	123	68	95.5	31849	916
S1196	112	10	61	4586	795
S1238	98	40	69	4589	698
S1488	102	6	54	10258	2854
S1494	119	11	65	12859	1425
S5378	545	22	283.5	18548	1922
S9234	1123	30	576.5	25866	3596
S13207	1659	42	850.5	40287	4987
S15850	2102	18	1060	39854	7584

Table 6: Comparison on consistency rate between GA and HTGA

Benchmark	Consistency rate	
Circuit	Genetic Algorithm	HTGA
S27	40	60
S208	46	63
S298	52	68
S832	58	66.25
S386	62.5	71
S641	48	62
S953	46	63
S1196	48	69.65
S1238	40.5	70.6
S1488	45.26	69.24
S1494	49.65	65
S5378	55	70.65
S9234	48.4	67.25
S13207	59.65	69
S15850	51	68.6

Table 7: Cutsize Comparison of HTGA with GA and TS (S.MSait)

Benchmark	Cutsize of the Benchmark Circuits		
Circuit	Genetic Algorithm	Tabu Search	HTGA
S298	19	24	13
S832	45	50	39
S386	36	30	32
S641	45	59	44
S953	96	99	84
S1196	123	106	102
S1238	127	79	73
S1488	104	98	92
S1494	102	101	101
S5378	573	430	463
S9234	1090	918	915
S13207	1683	1332	1328
S15850	2183	1671	1665

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Negation Switching Equivalence in Signed Graphs

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Abstract: A *Smarandachely k-signed graph* (*Smarandachely k-marked graph*) is an ordered pair $S = (G, \sigma)$ $(S = (G, \mu))$ where $G = (V, E)$ is a graph called *underlying graph of* S and $\sigma : E \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k)$ $(\mu : V \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k))$ is a function, where each $\overline{e}_i \in \{+, -\}$. Particularly, a Smarandachely 2-signed graph or Smarandachely 2-marked graph is called abbreviated a *signed graph* or a *marked graph*. In this paper, we establish a new graph equation $L^2(G) \cong L^k(G)$, where $L^2(G)$ & $L^k(G)$ are second iterated line graph and k^{th} iterated line graph respectively. Further, we characterize signed graphs S for which $L^2(S) \sim L^k(S)$ and $\eta(S) \sim L^k(S)$, where $\sim$ denotes switching equivalence and $L^2(S)$, $L^k(S)$ and $\eta(S)$ are denotes the second iterated line signed graph, k^{th} iterated line signed graph and negation of S respectively.

Key Words: Smarandachely k-signed graphs, Smarandachely k-marked graphs, signed graphs, marked graphs, balance, switching, line signed graphs, negation.

AMS(2000): 05C22

§1. Introduction

Unless mentioned or defined otherwise, for all terminology and notion in graph theory the reader is refer to [8]. We consider only finite, simple graphs free from self-loops.

A *Smarandachely k-signed graph* (*Smarandachely k-marked graph*) is an ordered pair $S = (G, \sigma)$ $(S = (G, \mu))$ where $G = (V, E)$ is a graph called *underlying graph of* S and $\sigma : E \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k)$ $(\mu : V \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k))$ is a function, where each $\overline{e}_i \in \{+, -\}$. Particularly, a Smarandachely 2-signed graph or Smarandachely 2-marked graph is called abbreviated a *signed graph* or a *marked graph*. Cartwright and Harary [5] considered graphs in which vertices represent persons and the edges represent symmetric dyadic relations amongst persons each of which designated as being positive or negative according to whether the nature of the relationship is positive (friendly, like, etc.) or negative (hostile, dislike, etc.). Such a network

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S is called a *signed graph* (Chartrand [6]; Harary et al. [11]).

Signed graphs are much studied in literature because of their extensive use in modeling a variety socio-psychological process (e.g., see Katai and Iwai [14], Roberts [16] and Roberts and Xu [17]) and also because of their interesting connections with many classical mathematical systems (Zaslavsky [25]).

A cycle in a signed graph S is said to be *positive* if the product of signs of its edges is positive. A cycle which is not positive is said to be *negative*. A signed graph is then said to be *balanced* if every cycle in it is positive (Harary [9]). Harary and Kabell [12] developed a simple algorithm to detect balance in signed graphs as also enumerated them.

A *marking* of S is a function $\mu : V(G) \to \{+,-\}$; A signed graph S together with a marking μ is denoted by S_μ. Given a signed graph S one can easily define a marking μ of S as follows: For any vertex $v \in V(S)$,

$$\mu(v) = \prod_{uv \in E(S)} \sigma(uv),$$

the marking μ of S is called *canonical marking* of S.

The following characterization of balanced signed graphs is well known.

Theorem 1(E. Sampathkumar, [18]) *A signed graph $S = (G, \sigma)$ is balanced if, and only if, there exists a marking μ of its vertices such that each edge uv in S satisfies $\sigma(uv) = \mu(u)\mu(v)$.*

The idea of switching a signed graph was introduced in [1] in connection with structural analysis of social behavior and also its deeper mathematical aspects, significance and connections may be found in [25].

Switching S with respect to a marking μ is the operation of changing the sign of every edge of S to its opposite whenever its end vertices are of opposite signs. The signed graph obtained in this way is denoted by $\mathcal{S}_\mu(S)$ and is called *μ-switched signed graph* or just *switched signed graph*. Two signed graphs $S_1 = (G, \sigma)$ and $S_2 = (G', \sigma')$ are said to be *isomorphic*, written as $S_1 \cong S_2$ if there exists a graph isomorphism $f : G \to G'$ (that is a bijection $f : V(G) \to V(G')$ such that if uv is an edge in G then $f(u)f(v)$ is an edge in G') such that for any edge $e \in G$, $\sigma(e) = \sigma'(f(e))$. Further a signed graph $S_1 = (G, \sigma)$ *switches* to a signed graph $S_2 = (G', \sigma')$ (or that S_1 and S_2 are *switching equivalent*) written $S_1 \sim S_2$, whenever there exists a marking μ of S_1 such that $\mathcal{S}_\mu(S_1) \cong S_2$. Note that $S_1 \sim S_2$ implies that $G \cong G'$, since the definition of switching does not involve change of adjacencies in the underlying graphs of the respective signed graphs.

Two signed graphs $S_1 = (G, \sigma)$ and $S_2 = (G', \sigma')$ are said to be *weakly isomorphic* (see [23]) or *cycle isomorphic* (see [23]) if there exists an isomorphism $\phi : G \to G'$ such that the sign of every cycle Z in S_1 equals to the sign of $\phi(Z)$ in S_2. The following result is well known (See [24]).

Theorem 2(T. Zaslavsky, [24]) *Two signed graphs S_1 and S_2 with the same underlying graph are switching equivalent if, and only if, they are cycle isomorphic.*

§2. Negation switching equivalence in signed graphs

One of the important operations on signed graphs involves changing signs of their edges. From socio-psychological point of view, if a signed graph represents the structure of a social system in which vertices represent persons in a social group, edges represent their pair-wise (dyadic) interactions and sign on each edge represents the qualitative nature of interaction between the corresponding members in the dyad classified as being positive or negative then according to social balance theory, the social system is defined to be in a balanced state if every cycle in the signed graph contains an even number of negative edges [9]; otherwise, the social system is said to be in an unbalanced state. The term balance used here refers to the real-life situation in which the individuals in a social group experience a state of cognitive stability in the sense that there is no psychological tension amongst them that demands a change in the pattern of their ongoing in- terpersonal interactions. For instance, as pointed out by Heider [13], any situation in which a person is forced to maintain a positive relation simultaneously with two other persons who are in conflict with each other is an unbalanced state of the triad consisting of the three persons. Thus, when the social system is found to be in an unbalanced state it is desired to bring it into a balanced state by means of forcing some positive (negative) relationships change into negative (positive) relationships; such sets of edges in the corresponding signed graph model are called balancing sets (see Katai & Iwai [14]). Such a change (which may be regarded as a unary operation transforming the given signed graph) is called *negation*, which has other implications in social psychology too (see Acharya & Joshi [2]). Thus, formally, the negation $\eta(S)$ of S is a signed graph obtained from S by negating the sign of every edge of S; that is, by changing the sign of each edge to its opposite [10].

Behzad and Chartrand [4] introduced the notion of line signed graph $L(S)$ of a given signed graph S as follows: Given a signed graph $S = (G, \sigma)$ its *line signed graph* $L(S) = (L(G), \sigma')$ is the signed graph whose underlying graph is $L(G)$, the line graph of G, where for any edge e_ie_j in $L(S)$, $\sigma'(e_ie_j)$ is negative if, and only if, both e_i and e_j are adjacent negative edges in S. Another notion of line signed graph introduced in [7] is as follows: The *line signed graph* of a signed graph $S = (G, \sigma)$ is a signed graph $L(S) = (L(G), \sigma')$, where for any edge ee' in $L(S)$, $\sigma'(ee') = \sigma(e)\sigma(e')$. In this paper, we follow the notion of line signed graph defined by M. K. Gill [7] (See also E. Sampathkumar et al. [19,20]).

Theorem 3(M. Acharya, [3]) *For any signed graph $S = (G, \sigma)$, its line signed graph $L(S) = (L(G), \sigma')$ is balanced.*

Hence, we shall call a given signed graph S a *line signed graph* if it is isomorphic to the line signed graph $L(S')$ of some signed graph S'. In [20], the authors obtained a structural characterization of line signed graphs as well as line signed digraphs.

For any positive integer k, the k^{th} iterated line graph, $L^k(G)$ of G (k^{th} iterated line signed graph, $L^k(S)$ of S) is defined as follows:

$$L^0(G) = G,\ L^k(G) = L(L^{k-1}(G))\ (L^0(S) = S,\ L^k(S) = L(L^{k-1}(S)))$$

Corollary 4(P. Siva Kota Reddy & M. S. Subramanya, [22]) *For any signed graph $S = (G, \sigma)$*

and for any positive integer k, $L^k(S)$ is balanced.

The following result is well known.

Theorem 5(V. V. Menon, [15]) *For a graph G, $G \cong L^k(G)$ for any integers $k \geq 1$ if, and only if, G is 2-regular.*

Proposition 6(D. Sinha, [21]) *For a connected graph $G = (V, E)$, $L(G) \cong L^2(G)$ if, and only if, G is cycle or $K_{1,3}$.*

From the above results we have the following result for graphs.

Theorem 7 *For any graph G, $L^2(G) \cong L^k(G)$ for some $k \geq 3$, if, and only if, G is either a cycle or $K_{1,3}$.*

Proof Suppose that $L^2(G) \cong L^k(G)$ for some $k \geq 3$. We observe that $L^k(G) = L^{k-2}(L^2(G))$. Hence, by Proposition 6, $L^2(G)$ must be a cycle. But for any graph G, $L(G)$ is a cycle if, and only if, G is either cycle or $K_{1,3}$. Since $K_{1,3}$ is a forbidden to line graph and $L(G)$ is a line graph, $G \neq K_{1,3}$. Hence $L(G)$ must be a cycle. Finally $L(G)$ is a cycle if, and only if, G is either a cycle or $K_{1,3}$.

Conversely, if G is a cycle C_r , of length r, $r \geq 3$ then for any $k \geq 2$, $L^k(G)$ is a cycle and if $G = K_{1,3}$ then for any $k \geq 2$, $L^k(G) = C_3$. This implies, $L^2(G) = L^k(G)$ for any $k \geq 3$. This completes the proof. □

We now characterize those second iterated line signed graphs that are switching equivalent to their k^{th} iterated line signed graphs.

Proposition 8 *For any signed graph $S = (G, \sigma)$, $L^2(S) \sim L^k(S)$ if, and only if, G is either a cycle or $K_{1,3}$.*

Proof Suppose $L^2(S) \sim L^k(S)$. This implies, $L^2(G) \cong L^k(G)$ and hence by Theorem 7, we see that the graph G must be isomorphic to either a cycle or $K_{1,3}$.

Conversely, suppose that G is a cycle or $K_{1,3}$. Then $L^2(G) \cong L^k(G)$ by Theorem 7. Now, if S any signed graph on any of these graphs, By Corollary 4, $L^2(S)$ and $L^k(S)$ are balanced and hence, the result follows from Theorem 2. □

We now characterize those negation signed graphs that are switching equivalent to their line signed graphs.

Proposition 9 *For any signed graph $S = (G, \sigma)$, $\eta(S) \sim L^k(S)$ if, and only if, S is an unbalanced signed graph and G is 2-regular with odd length.*

Proof Suppose $\eta(S) \sim L^k(S)$. This implies, $G \cong L^k(G)$ and hence G is 2-regular. Now, if S is any signed graph with underlying graph as 2-regular, Corollary 4 implies that $L^k(S)$ is balanced. Now if S is an unbalanced signed graph with underlying graph $G = C_n$, where n is even, then clearly $\eta(S)$ is unbalanced. Next, if S is unbalanced signed graph with underlying graph $G = C_n$, where n is odd, then $\eta(S)$ is unbalanced. Hence, if $\eta(S)$ is unbalanced and

its line signed graph $L^k(S)$ being balanced can not be switching equivalent to S in accordance with Theorem 2. Therefore, S must be unbalanced and G is 2-regular with odd length.

Conversely, suppose that S is an unbalanced signed graph and G is 2-regular with odd length. Then, since $L^k(S)$ is balanced as per Corollary 4 and since $G \cong L^k(G)$, the result follows from Theorem 2 again. $\square$

Corollary 10 *For any signed graph $S = (G, \sigma)$, $\eta(S) \sim L(S)$ if, and only if, S is an unbalanced signed graph and G is 2-regular with odd length.*

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Weak and Strong Reinforcement Number For a Graph

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Abstract: Let G=(V(G),E(G)) be a graph.A set of vertices S in a graph G is called to be a Smarandachely dominating k-set, if each vertex of G is dominated by at least k vertices of S. Particularly, if $k = 1$, such a set is called a dominating set of G. The Smarandachely domination k-number $\gamma_k(G)$ of G is the minimum cardinality of a Smarandachely dominating k-set of G. S is called weak domination set if $deg(u) \leq deg(v)$ for every pair of $(u, v) \in V(G) - S$. The minimum cardinality of a weak domination set S is called weak domination number and denoted by $\gamma_w(G)$. In this paper we introduce the weak reinforcement number which is the minimum number of added edges to reduce the weak dominating number. We give some boundary of this new parameter and trees. Furthermore, some boundary of strong reinforcement number has been given for a given graph G and its complemented graph $\overline{G}$.

Key Words: Connectivity, Smarandachely dominating k-set,Smarandachely dominating k-number, strong or weak reinforcement number.

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§1. Introduction

Let $G = (V, E)$ be a graph with vertex set V and edge set E. A set $S \subseteq V$ is a Smarandachely dominating k-set of G if every vertex v in $V - S$ there exists a vertex u in S such that u and v are adjacent in G. The Smarandachely domination k-number of G, denoted $\gamma_k(G)$ is the minimum cardinality of a Smarandachely dominating k-set of G [7]. The concept of dominationin graphs, with its many variations, is well studied in graph theory and also many kind of dominating k-numbers have been described. Strong domination(sd-set) and weak domination(sw-set) was introduced by Sampathkumar and Latha [2]. Let $uv \in E$. Then u and v dominate each other. Further, u strongly dominates [weakly dominates] v if $deg(u) \geq deg(v)$ [$deg(u) \leq deg(v)$]. A set $S \subseteq V$ is strong dominating set(sd-set) [weakly dominating set(sw-set)] if every vertex $v \in V - S$ is strongly dominated [weakly dominated] by some u in S. The strong domination number $\gamma_s(G)$ [weak domination number $\gamma_w(G)$] of G is the minimum cardinality of a Smarandachely dominating k-set S [5]. If Smarandachely domination k-number of G is small, then distance between each pair of vertices is small in G. This property is easily see that

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$\gamma_k(G) = \gamma_s(G) = \gamma_w(G) = 1$, where G is complete and the distance between each pairs is 1. If any edge could removed from graph G then the Smarandachely domination k-number of G increase. Fink et al.[4] introduced the bondage number of a graph in 1990. The bondage number $b(G)$ of a nonempty graph G is the cardinality of a smallest set of edges whose removal from G results in a graph with Smarandachely domination k-number grater than $\gamma_k(G)$ [1,4,5]. Strong and weak bondage number introduced by Ebadi et al. in 2009 [7]. If some edge added from graph G then the Smarandachely domination k-number of G could decrease. In 1990, Kok and Mynhardt [6] introduced the reinforcement number $r(G)$ of a graph G, which is the minimum number of extra edges whose addition to graph G results in a graph G' with $\gamma_k(G) < \gamma_k(G')$. They defined $r(G) = 0$ if $\gamma_k(G) = 1$. In 1995, Ghoshol et al. introduced strong reinforcement number r_s , the cardinality of a smallest set F which satisfies $\gamma_s(G+F) < \gamma_s(G)$ where $F \subset E(\overline{G})$ [5]. In Figure1, $\gamma_k(G) = 2$,$\gamma_s(G) = 3$,$\gamma_w(G) = 4$, $r(G) = 2$ and $r_s(G) = 1$ for graph G.

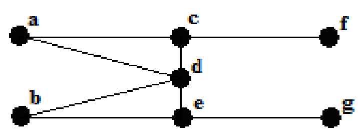

Figure1: Graph G

Cardinality of $\{c, e\}$-set equals to the $\gamma_k(G)$, cardinality of $\{c, d, e\}$-set equals to the $\gamma_s(G)$, cardinality of $\{a, b, f, g\}$-set equals to the $\gamma_w(G)$. Moreover, when we add two edge from vertex d to vertex f and g, $\gamma_k(G)$ decrease. Then,$r(G) = 2$. Similarly, when we add an edge from vertex c to vertex g, it is easy to see that $r_s(G) = 1$. In this paper, for $\Delta(G)$ and $\delta(G)$ denote the number of maximum and minimum degree, respectively.

§2 Weak reinforcement number

In this section we introduced a new reinforcement concept. Let F be a subset of $E(\overline{G})$. Weak reinforcement number r_w , the cardinality of smallest set F which satisfies$\gamma_w(G+F) < \gamma_w(G)$. Then here, some weak reinforcement number boundaries' are given and reinforcement numbers of basic graph are computed.

Theorem 2.1 *Let* G *be a connected graph, then* $1 \leq r_w \leq \frac{n.(n-1)}{2} - m$ *, where* $n = |V(G)|$ *and* $m = |E(G)|$ *for any graph* G*.*

Proof If $\Delta(G) = n - 1$, then $r_w(G) = 0$ by definition. To dominate all vertices of a graph by a vertex which has minimum degree, it is necessary all vertices have $n - 1$ degree, so the graph is a complete graph. For any graph G, we can add $\frac{n.(n-1)}{2} - m$ edges to make a complete graph and it's an upper boundary. Lower boundary is 1, because of star graph's structure. Consequently, when we add at least 1-edge and at most $\frac{n.(n-1)}{2} - m$, decrease $\gamma_w(G)$. □

Observation 2.1 If G is a complete graph then,$\gamma_w(G) = 1$.

Theorem 2.2 *If $\gamma_w(G)$ is 2, then $r_w(G) = \frac{n.(n-1)}{2} - m$ for any graph* G.

Proof Let weak domination number of a graph G be 2. We can decrease this number only 1. Due to the Observation 2.1 the graph G must be a complete. To make graph G complete must add $|E(\overline{G})|$ edges to graph, i.e. we must add $\frac{n.(n-1)}{2} - m$ edges. □

Lemma 2.1([6]) *The weak and strong domination number of* n-cycle *is*

$$\gamma_w(C_n) = \gamma_s(C_n) = \lceil \frac{n}{3} \rceil for, n \geq 3.$$

Theorem 2.3 *The weak reinforcement number of the* n-cycle *(with $n \geq 7$ and $n \neq 9$) is*

$$r_w(C_n) = \begin{cases} 2\,, \ n = 1 \ (mod3) \\ 4\,, \ n = 2 \ (mod3) \\ 6\,, \ n = 0 \ (mod3) \end{cases}$$

Proof From Lemma 2.1, the weak domination number of graph C_n is $\lceil \frac{n}{3} \rceil$. When $\gamma_w(G)$ is decreased, there arises 3 cases.

Case 1 If $n \equiv 1(mod3)$, the vertex which is taken to weak domination set, including itself dominates 3 vertices. In order for a vertex to dominate both itself and the other 3 vertices, to graph C_n two edges are added (see Figure 2).

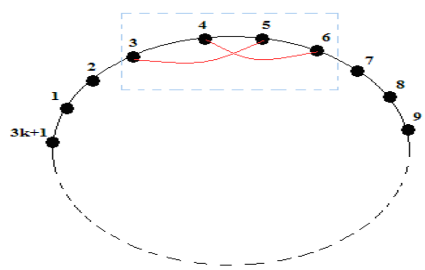

Figure 2

In conclusion, in the weak domination set there are vertices from K_4 structure in Figure 2 together with the $\frac{n-4}{3}$ vertices. Then,$\gamma_w(C_n + F) = \frac{n-4}{3} + 1 = \frac{n-1}{3}$. Since $\frac{n-1}{3} < \lceil \frac{n}{3} \rceil$, then $r_w = 2$.

Case 2 If $n \equiv 2(mod3)$, similar to Case1, by creating two K_4 structure, the proof is set. In conclusion, in the weak domination set there has been $\frac{n-8}{3}$+2 vertices. Then,$\gamma_w(C_n+F) = \frac{n-2}{3}$. Since $\frac{n-2}{3} < \lceil \frac{n}{3} \rceil$, then $r_w = 4$.

Case 3 If $n \equiv 0(mod3)$, when it is set similar to Case1,$r_w = 6$.

Combining Cases 1-3, the proof is complete. □

Theorem 2.4 *Values of weak reinforcement number of C_4,C_5,C_6 and C_9 are 2, 5, 9 and 7, respectively.*

Proof The weak reinforcement number of C_4,C_5 and C_6 are 2. It is easily seeing that from Theorem 2.2, $r_w(C_4) = 2$,$r_w(C_5) = 2$,$r_w(C_6) = 9$. Moreover, $\gamma_w(C_9) = 3$. To decrease this number,we must obtain a K_4 and K_5 from C_9 vertices.Then it's easily see that $r_w(C_9) = 7$. □

Lemma 2.2([4]) *The weak and strong domination number of the path of* order-n *is*

$$\gamma_s(P_n) = \lceil \frac{n}{3} \rceil, forn \geq 3,$$

$$\gamma_w(P_n) = \begin{cases} \lceil \frac{n}{3} \rceil & , \ if \ n = 1 \ (mod 3), \\ \lceil \frac{n}{3} \rceil + 1 & , \ otherwise \end{cases}$$

Theorem 2.5 *The weak reinforcement number of the* path of order-n *is*

$$r_w(P_n) = \begin{cases} 3 \, , \ n = 1 \ (mod 3) \\ 1 \, , \ otherwise. \end{cases}$$

Proof If $n = 3k$ and $n = 3k+2$ then $\gamma_w(P_n) = \lceil \frac{n}{3} \rceil + 1$. For these cases, we add an e_1-edge to two vertices, which has degree 1, then the graph be a C_n. $\gamma_w(C_n) > \gamma_w(C_n + e_1)$ since $\gamma_w(C_n)$ is $\lceil \frac{n}{3} \rceil$. For this reason, $r_w(P_n) = 1$. If $n = 3k + 1$ then we add an edge to two vertices, which has degree 1, then the graph be a C_n. Then we add 2 more edges, likes Theorem 2.3, Case1. Since $\gamma_w(C_n) > \gamma_w(C_n + F)$, then $r_w(P_n) = 3$, where F is a set of added edges. □

Lemma 2.3([4]) *The weak and strong domination number of the wheel graph $W_{1,n}$ is*

$$\gamma_s(W_{1,n}) = 1 \ , \ \gamma_w(W_{1,n}) = \lceil \frac{n}{3} \rceil.$$

Theorem 2.6 *The weak reinforcement number of the wheel graph $W_{1,n}$ (with $n \geq 7$ and $n \neq 9$)*

$$r_w(W_{1,n}) = \begin{cases} 2 \, , \ n = 1 \ (mod 3) \\ 4 \, , \ n = 2 \ (mod 3) \\ 6 \, , \ n = 0 \ (mod 3) \end{cases}$$

Proof The proof is similar to that of Theorem 2.3. □

Theorem 2.7 *If $n = 4, 5, 6, 9$ then $r_w(W_{1,n})$ is 2,5,9 and 7, respectively.*

Proof The proof makes similar to that of Theorem 2.4. □

Lemma 2.4([5]) *The weak and strong domination number of the complete bipartite graph $K_{m,n}$ is*

$$\gamma_s(K_{m,n}) = \begin{cases} 2 & , \ if \ 2 \leq m = n, \\ m & , \ if \ 1 \leq m < n. \end{cases}$$

$$\gamma_w(K_{m,n}) = \begin{cases} 2 & , \ if \ 2 \leq m = n, \\ n & , \ if \ 1 \leq m < n. \end{cases}$$

Theorem 2.8 *The weak reinforcement number of complete bipartite graph $K_{m,n}$, where $m \leq n$ is*

$$r_w(K_{m,n}) = \begin{cases} m^2 - m & , \quad m = n \geq 2, \\ 1 & , \quad m < n. \end{cases}$$

Proof If $m = n$, then $\gamma_w(K_{m,n}) = 2$. Due to Theorem 2.2, the graph must be a complete while weak domination number decreasing. The edge number of graph K_{2m} is $\dfrac{2m(2m-1)}{2}$. The edge number of $K_{m,n}$ is m^2. So, r_w number is $m^2 - m$. If $m < n$ then $\gamma_w(K_{m,n}) = n$. When we add an edge between two vertices which have degree of m, we obtain the r_w number is 1. □

Result 2.1 *If m=1, then $r_w(K_{1,n}) = 1$, where $K_{1,n}$ is a star graph.*

Lemma 2.5([5]) *Define a support to be a vertex in a tree which adjacent to an end-vertex. Every tree T with $(n \geq 4)$ has at least one of the following characteristic.*

(*i*) *A support adjacent to at least 2 end-vertex;*
(*ii*) *A support is adjacent to a support of degree 2;*
(*iii*) *A vertex is adjacent to 2 support of degree 2;*
(*iv*) *A support of a leaf and the vertex adjacent to the support are both of degree 2.*

Theorem 2.9 *If any vertex of tree T is adjacent with two or more end-vertices, then $r_w(T) = 1$.*

Proof Let T has two or more end-vertices, which denote by $u_1, u_2, \ldots$. The u_i's belong to every minimum weak domination set of T. We add an e-edge between two vertices, then $\gamma_w(T) > \gamma_w(T + e)$. Hence, $r_w(T) = 1$. □

Theorem 2.10 *Let T be any tree order of n $(n > 3)$, then $r_w(T) \leq 3$.*

Proof It is easy to see that $r_w(T) = 0$ and $r_w(T) = 1$ for n=2 and n=3, respectively. Assume that $n > 3$. From Lemma 2.5, there are 4 cases. (see Figure3)

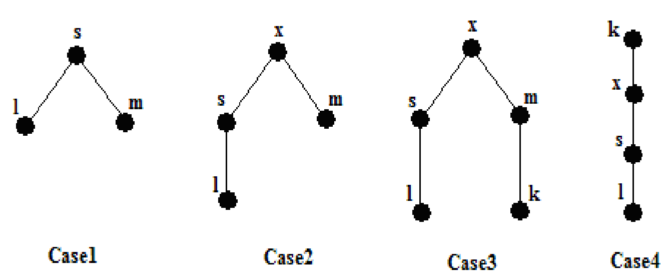

Figure3: End characteristic of trees

Case 1 Assume that supported vertex s is adjacent to two or more vertices. All end-vertices are in weak domination set. When we add an e-edge between any two end-vertices, $\gamma_w(T) > \gamma_w(T + e)$ is obtained. Hence, $r_w(T) = 1$.

Case 2 In this case two end-vertices are in weak domination set. We must obtain K_4 structure for weak dominate four vertices by a vertex. For this, worst case situation, we must add three edges. Hence,$r_w(T) \leq 3$.

Case 3 **and Case** 4 The proofs make similar to Case2. Consequently, $r_w(T) \leq 3$.

Combining Cases 1-4, the proof is complete. □

§3. Strong reinforcement number

In these section general results is given for strong reinforcement number and some boundaries of strong reinforcement number of any graph G and its complemented graph $\overline{G}$. In [5], Theorems 3.1-3.6 following are proved.

Theorem 3.1 *The strong reinforcement number of the* path of order-n *and* n-cycle *is*

$$r_s(P_n) = r_s(C_n) = i, where n \equiv i(mod 3).$$

Theorem 3.2 *The strong reinforcement number of multipicle complete graph is*

$$r_s(K_{m_1,m_2,\ldots,m_t}) = \begin{cases} 0 & , if\, m_1 = 1 \\ m_1 - 1 & , if\, m_1 \neq 1\, and\, m_1 = m_2 \\ 1 & , if\, m_1 \neq 1\, and\, m_1 \neq m_2 \end{cases}$$

Theorem 3.3 $r_s(G) \leq p - 1 - \Delta(G)$ *for any graph* G, *where* $p = |V(G)|$.

Theorem 3.4 *If* G *is any graph* G, *then* $r_s(G) = p - 1 - \Delta(G)$ *if and only if* $\gamma_s(G) = 2$.

Theorem 3.5 $r_s(G) \leq \Delta(G) + 1$, *for any graph* G *with* $\gamma_s(G) \geq 2$.

Theorem 3.6 $\gamma_s(G) + r_s(G) \leq p - \Delta(G) + 1$ *for any graph G, where* $p = |V(G)|$.

Theorem 3.7 *Let* G *be any graph* order of n *and* $\overline{G}$ *be a complemented graph of* G. *If graph* G *has at least one vertex which has degree 1, then* $\gamma_s(\overline{G}) = 2$ *and* $r_s(\overline{G}) = 1$.

Proof Let vertex u has degree 1. vertex u adjacent to *n-2* vertices in $\overline{G}$. Then taking vertex v in strong domination set where vertex v adjacent to vertex u. Hence,$\gamma_s(\overline{G}) = 2$ .From Theorem 3.4,$r_s(\overline{G}) = p - 1 - \Delta(\overline{G})$. Since $\Delta(\overline{G}) = n - 2$, it is easily see that $r_s(\overline{G}) = 1$. □

Theorem 3.8 *Let G be any graph* order of n *and* $\overline{G}$ *be a complemented graph of* G. *Then,* $r_s(\overline{G}) \leq \delta(G)$.

Proof It is obvious that $\Delta(\overline{G}) = n - \delta(G) - 1$ and $r_s(\overline{G}) \leq n - 1 - \Delta(\overline{G})$ from the Theorem 3.3. Whence,

$$r_s(\overline{G}) \leq n - 1 - (n - \delta(G) - 1), \quad r_s(\overline{G}) \leq \delta(G).$$ □

Theorem 3.9 *Let* G *be any graph* order of n *and* $\overline{G}$ *be a complemented graph of* G. *Then,* $r_s(G) + r_s(\overline{G}) \leq n + \delta(G) - (\Delta(G) + 1)$.

Proof It easily see that from Theorems 3.3 and 3.8. □

§4. Conclusion

The concept of domination in graph is very effective both in theoretical developments and applications. Also, domination set problem can be used to solve hierarchy problem, network flows and many combinatoric problems. If graph G has a small domination number then each pairs of vertex has small distance. So, in any graph if we want to decrease to domination number, we have to decrease distance between each pairs of vertex. More than thirty domination parameters have been investigated by different authors, and in this paper we have introduced the concept of domination. We called weak reinforcement number its. Then, we computed weak reinforcement number for some graph and some boundary of strong reinforcement number has been given for a given graph G and its complemented graph $\overline{G}$. Work on other domination parameters will be reported in forthcoming papers.

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Tulgeity of Line, Middle and Total Graph of Wheel Graph Families

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Abstract: Tulgeity $\tau(G)$ is the maximum number of disjoint, point induced, non acyclic subgraphs contained in G. In this paper we find the tulgeity of line, middle and total graph of wheel graph, Gear graph and Helm graph.

Key Words: Tulgeity, Smarandache partition, line graph, middle graph, total graph and wheel graph.

AMS(2000): 05C70, 05C75, 05C76

§1. Introduction

The *point partition number* [4] of a graph G is the minimum number of subsets into which the point-set of G can be partitioned so that the subgraph induced by each subset has a property P. Dual to this concept of point partition number of graph is the maximum number of subsets into which the point-set of G can be partitioned such that the subgraph induced by each subset does not have the property P. Define the property P such that a graph G has the property P if G contains no subgraph which is homeomorphic to the complete graph K_3. Now the point partition number and dual point partition number for the property P is referred to as point arboricity and tulgeity of G respectively. Equivalently the tulgeity is the maximum number of vertex disjoint subgraphs contained in G so that each subgraph is not acyclic. This number is called the tulgeity of G denoted by $\tau(G)$. Also, $\tau(G)$ can be defined as the maximum number

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of disjoint cycles in G. The formula for tulgeity of a complete bipartite graph is given in [5]. The problems of Nordhaus-Gaddum type for the dual point partition number are investigated in [3].

Let P be a graph property and G be a graph. If there exists a partition of G with a partition set pair $\{H,T\}$ such that the subgraph induced by a subset in H has property P, but the subgraph induced in T has no property P, then we say G possesses the *Smarandache partition*. Particularly, let $H=\emptyset$ or $T=\emptyset$, we get the conception of point partition or its dual.

All graphs considered in this paper are finite and contains no loops and no multiple edges. Denote by $[x]$ the greatest integer less than or equal to x, by $|S|$ the cardinality of the set S, by $E(G)$ the edge set of G and by K_n the complete graph on n vertices. p_G and q_G denotes the number of vertices and edges of the graph G. The other notations and terminology used in this paper can be found in [6].

Line graph $L(G)$ of a graph G is defined with the vertex set $E(G)$, in which two vertices are adjacent if and only if the corresponding edges are adjacent in G. Since $\tau(G) \leq \left[\frac{p}{3}\right]$, it is obvious that $\tau(L(G)) \leq \left[\frac{q}{3}\right]$. However for complete graph K_p, $\tau(K_p) = \left[\frac{p}{3}\right]$.

Middle graph $M(G)$ of a graph G is defined with the vertex set $V(G)\cup E(G)$, in which two elements are adjacent if and only if either both are adjacent edges in G or one of the elements is a vertex and the other one is an edge incident to the vertex in G. Clearly $\tau(M(G)) \leq \left[\frac{p+q}{3}\right]$.

Total graph $T(G)$ of a graph G defined with the vertex set $V(G)\cup E(G)$, in which two elements are adjacent if and only if one of the following holds true (i) both are adjacent edges or vertices in G (ii) one is a vertex and other is an edge incident to it in G.

§2. Basic Results

We begin by presenting the results concerning the tulgeity of a graph.

Theorem 2.1([5]) *For any graph G, $\tau(G) = \sum \tau(C) \leq \tau(B)$, where the sums being taken over all components C and blocks B of G, respectively.*

Theorem 2.2([5]) *For the complete n-partite graph $G = K(p_1, p_2, ..., p_n), 1 \leq p_1 \leq p_2 \leq ..... \leq p_n$ and $\sum p_i = p$, $\tau(G) = min\left(\left[\frac{1}{2}\sum_0^{n-1} p_i\right], [p/3]\right)$, where $p_0 = 0$.*

We have derived [1] the formula to find the tulgeity of the line graph of complete and complete bigraph.

Theorem 2.3([1]) $\tau(L(K_n)) = \left[\frac{n(n-1)}{6}\right]$.

Theorem 2.4([1]) $\tau(L(K_{m,n}) = \left[\frac{mn}{3}\right]$.

Also, we have derived an upper bound for the tulgeity of line graph of any graph and characterized the graphs for which the upper bound equal to the tulgeity.

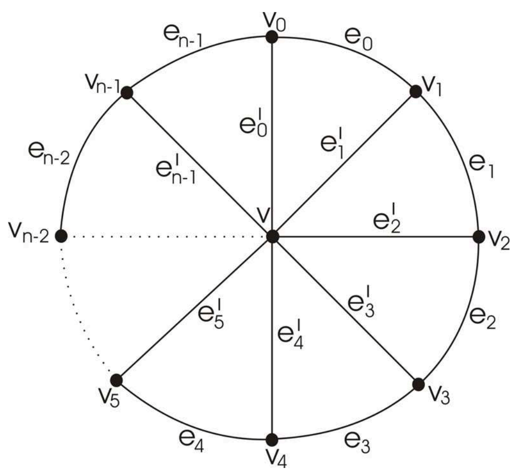
Theorem 2.5([1]) *For any graph G, $\tau(L(G)) \leq \sum_i \left[\frac{\deg v_i}{3}\right]$ where $\deg v_i$ denotes the degree of the vertex v_i and the the summation taken over all the vertices of G.*

Theorem 2.6([1]) *If G is a tree and for each pair of vertices (v_i, v_j) with $\deg v_i$, $\deg v_j > 2$, if there exist a vertex v of degree 2 on $P(v_i, v_j)$ then $\tau(L(G)) \leq \sum_i \left[\frac{\deg v_i}{3}\right]$.*

We have derived the results to find the tulgeity of Knödel graph, Prism graph and their line graph in [2].

§3. Wheel Graph

The wheel graph W_n on $n+1$ vertices is defined as $W_n = C_n + K_1$ where C_n is a n-cycle. Let $V(W_n) = \{v_i : 0 \leq i \leq n-1\} \cup \{v\}$ and $E(W_n) = \{e_i = v_i v_{i+1} : 0 \leq i \leq n-1, \text{subscripts modulo } n\} \cup \{e_i' = vv_i : 0 \leq i \leq n-1\}$.



Wheel graph W_n

Figure 3.1

Theorem 3.1 *The Tulgeity of the line graph of W_n,*

$$\tau(L(W_n)) = \left[\frac{2n}{3}\right].$$

Proof By the definition of line graph, $V(L(W_n)) = E(W_n) = \{e_i : 0 \leq i \leq n-1, \text{subscripts modulo n}\} \cup \{e_i' : 0 \leq i \leq n-1\}$. Let

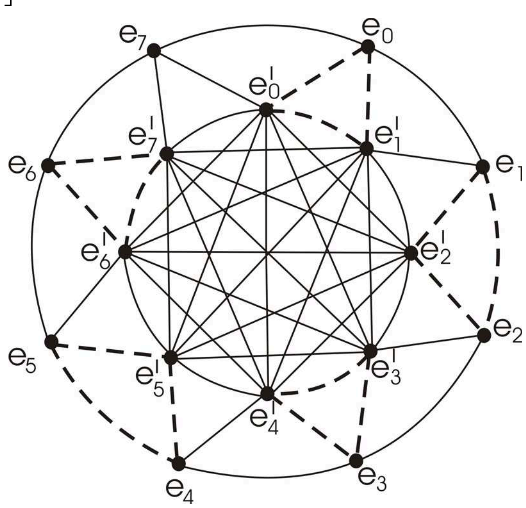
$$\mathbb{C} = \left\{e_i e_i' e_{i+1}' : i = 3(k-1), 1 \leq k \leq \left[\frac{n}{3}\right]\right\}$$

and

$$\mathbb{C}' = \left\{e_i e_{i+1} e_{i+1}' : i = 3k-2, 1 \leq k \leq \left[\frac{n}{3}\right]\right\}$$

be a collection of 3-cycles of $L(W_n)$. Clearly the cycles of $\mathbb{C}$ and $\mathbb{C}'$ are vertex disjoint and if $V(\mathbb{C})$ and $V(\mathbb{C}')$ denotes the set of vertices belonging to the cycles of $\mathbb{C}$ and $\mathbb{C}'$ respectively then $V(\mathbb{C}) \cap V(\mathbb{C}') = \emptyset$. Hence $\tau(L(W_n)) \geq |\mathbb{C}| + |\mathbb{C}'| = 2\left[\frac{n}{3}\right]$.

If $n \equiv 0$ or $1(mod\ 3)$, then $2\left[\frac{n}{3}\right] = \left[\frac{2n}{3}\right]$. Hence $\tau(L(W_n)) \geq \left[\frac{2n}{3}\right]$. If $n \equiv 2(mod\ 3)$, then $\left[\frac{2n}{3}\right] = 2\left[\frac{n}{3}\right] + 1$. In this case $e'_{n-2}, e'_{n-1}, e_{n-2}, e_{n-1} \notin V(\mathbb{C}) \cup V(\mathbb{C}')$ and the set $\{e'_{n-2}, e'_{n-1}, e_{n-2}\}$ induces a 3-cycle. Hence if $n \equiv 2(mod\ 3)$, $\tau(L(W_n)) \geq 2\left[\frac{n}{3}\right] + 1 = \left[\frac{2n}{3}\right]$. Therefore in both the cases $\tau(L(W_n)) \geq \left[\frac{2n}{3}\right]$. Also since $|V(L(W_n))| = 2n$, $\tau(L(W_n)) \leq \left[\frac{2n}{3}\right]$. Hence $\tau(L(W_n)) = \left[\frac{2n}{3}\right]$. □



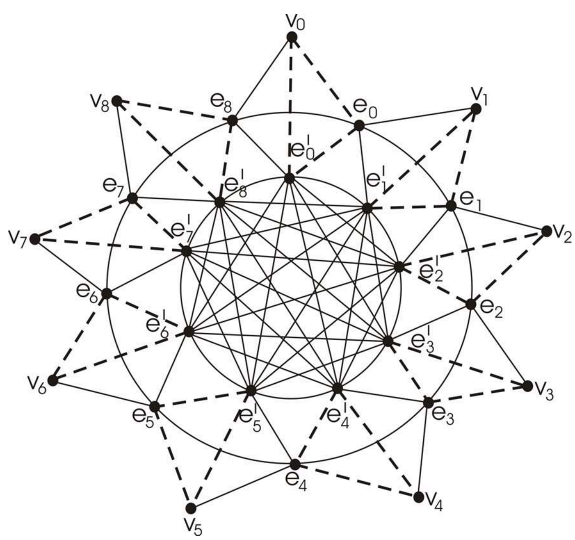
$L(W_8)$ and its vertex disjoint cycles

Figure 3.2

Theorem 3.2 *The Tulgeity of the middle graph of* W_n, $\tau(M(W_n)) = n$.

Proof By the definition of middle graph, $V(M(W_n)) = V(W_n) \cup E(W_n)$, in which for any two elements $x, y \in V(M(W_n))$, $xy \in E(M(W_n))$ if and only if any one of the following holds. (i) $x, y \in E(W_n)$ such that x and y are adjacent in W_n, (ii) $x \in V(W_n)$, $y \in E(W_n)$ or $x \in E(W_n)$, $y \in V(W_n)$ such that x and y are incident in W_n. Since $V(M(W_n)) = V(W_n) \cup E(W_n)$, $|V(M(W_n))| = n + 1 + 2n = 3n + 1$ and hence $\tau(M(W_n)) \leq \left[\frac{3n+1}{3}\right] = n$. Let $\mathbb{C} = \{C_i = v_i e_i e'_i : 0 \leq i \leq n-1\}$ be the collection of cycles of $M(W_n)$. Clearly the cycles of $\mathbb{C}$ are vertex disjoint and $|\mathbb{C}| = n$. Hence $\tau(M(W_n)) \geq n$ which implies $\tau(M(W_n)) = n$. □

By the definition of total graph $V(M(W_n)) = V(T(W_n))$ and $E(M(W_n)) \subset E(T(W_n))$. Also since $\tau(M(W_n)) = n = \left[\frac{1}{3} p_{M(W_n)}\right]$, we conclude the following result.



$M(W_9)$ and its vertex disjoint cycles

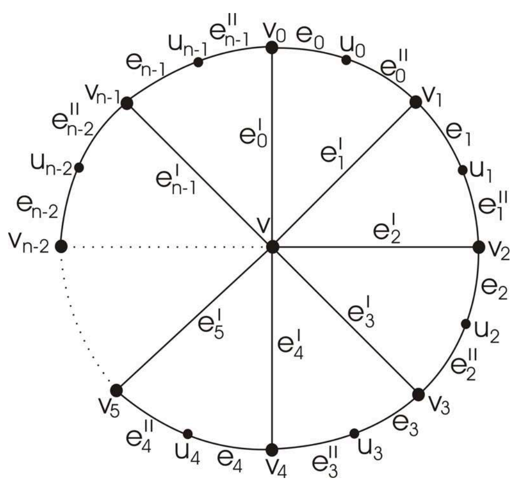
Figure 3.3

Theorem 3.3 *For any wheel graph W_n, the tulgeity of its total graph,*

$$\tau(T(W_n)) = \tau(M(W_n)) = n.$$

§4. Gear Graph

The gear graph is a wheel graph with vertices added between pair of vertices of the outer cycle. The gear graph G_n has $2n+1$ vertices and $3n$ edges.



Gear Graph G_n

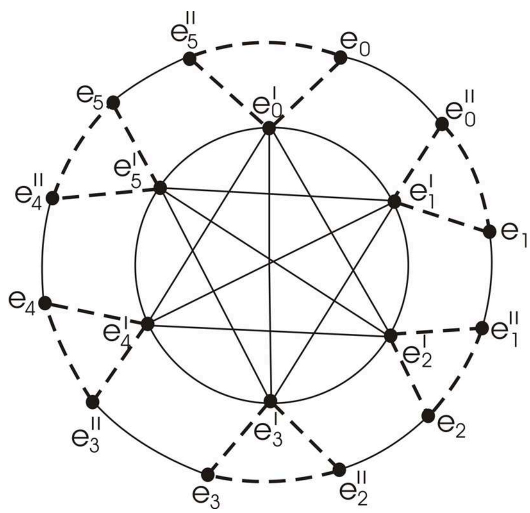
Figure 4.1

Let $V(G_n) = \{v_i : 0 \le i \le n-1\} \cup \{u_i : 0 \le i \le n-1\} \cup \{v\}$ and $E(G_n) = \{e_i = v_i u_i : 0 \le i \le n-1\} \cup \{e'_i = vv_i : 0 \le i \le n-1\} \cup \{e''_i = u_i v_{i+1} : 0 \le i \le n-1, \text{subscripts modulo } n\}$.

Theorem 4.1 *For any gear graph G_n, the tulgeity of its line graph,*

$$\tau(L(G_n)) = n.$$

Proof By the definition of line graph, $V(L(G_n)) = E(G_n)$, in which the set of vertices of $L(G_n)$, $\{e'_i : 0 \le i \le n-1\}$ induces a clique of order n. Also for each i, $(0 \le i \le n-1)$, the set $\{e''_i e'_{i+1} e_{i+1} : \text{subscripts modulo } n\}$ induces vertex disjoint clique of order 3. Let $\mathbb{C} = \{e''_i e'_{i+1} e_{i+1} : 0 \le i \le n-1, \text{subscripts modulo } n\}$ be the set of cycles of $L(G_n)$. It is clear that the cycles of $\mathbb{C}$ are vertex disjoint and $|\mathbb{C}| = n$ therefore $\tau(L(G_n)) \ge n$. Also, since $p_{L(G_n)} = q_{G_n} = 3n$, $\tau(L(G_n)) \le \left[\dfrac{3n}{3}\right] = n$. Hence $\tau(L(G_n)) = n$. □



$L(G_6)$ and its vertex disjoint cycles

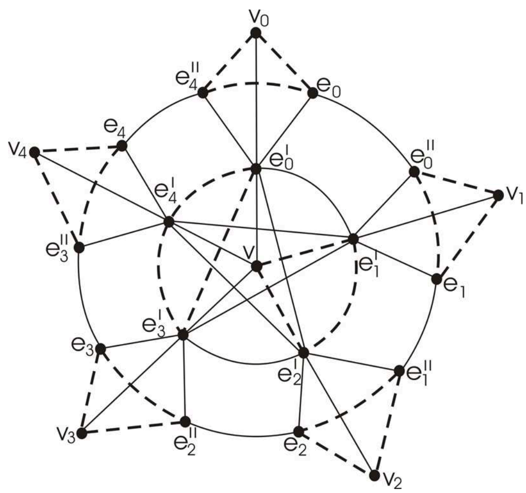
Figure 4.2

Theorem 4.2 *For any gear graph G_n, the tulgeity of its middle graph,*

$$\tau(M(G_n)) = \left[\frac{4n+1}{3}\right].$$

Proof Since $p_{M(G_n)} = p_{G_n} + q_{G_n} = (n+1) + 3n = 4n+1$, $\tau(M(G_n)) = \left[\dfrac{4n+1}{3}\right]$. By the definition of middle graph $V(M(G_n)) = V(G_n) \cup E(G_n)$, in which the set of vertices $\{e'_i : 0 \le i \le n-1\} \cup \{v\}$ induces a clique K_{n+1} of order $n+1$ and for each i, $(0 \le i \le n-1)$ the set $\{e''_i e'_{i+1} e_{i+1} v_{i+1} : \text{subscripts modulo } n\}$ induces a clique of order 4. From these cliques we form the set of cycles of $M(G_n)$. Let $\mathbb{C}$={set of vertex disjoint 3-cycles of the clique K_{n+1}} and $\mathbb{C}' = \{e''_i e'_{i+1} e_{i+1} v_{i+1} : 0 \le i \le n-1, \text{subscripts modulo } n\}$. Clearly $V(\mathbb{C}) \cap V(\mathbb{C}') = \emptyset$

and hence the cycles of $\mathbb{C}$ and $\mathbb{C}'$ are vertex disjoint. Also $|\mathbb{C}| = \left[\dfrac{n+1}{3}\right]$ and $|\mathbb{C}'| = n$. Hence $\tau(M(G_n)) \geq |\mathbb{C}| + |\mathbb{C}'| = \left[\dfrac{4n+1}{3}\right]$. Therefore $\tau(M(G_n)) = \left[\dfrac{4n+1}{3}\right]$. $\square$



$M(G_5)$ and its vertex disjoint cycles

Figure 4.3

By the definition of total graph $V(M(G_n)) = V(T(G_n))$ and $E(M(G_n)) \subset E(T(G_n))$. Also since $\tau(M(G_n)) = \left[\dfrac{4n+1}{3}\right] = \left[\dfrac{1}{3}p_{M(G_n)}\right]$, we conclude the following result.

Theorem 4.3 *For any gear graph G_n, the tulgeity of its middle graph,*

$$\tau(M(G_n)) = \tau(T(G_n)) = \left[\frac{4n+1}{3}\right].$$

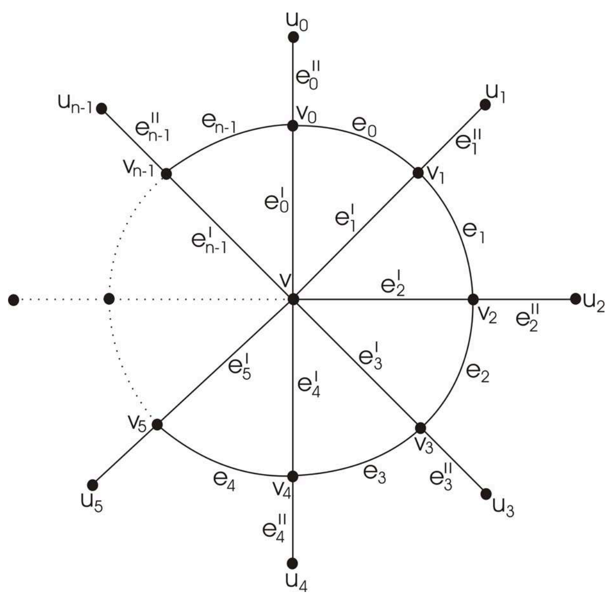
§5. Helm Graph

The helm graph H_n is the graph obtained from an n-wheel graph by adjoining a pendant edge at each node of the cycle.

Let $V(H_n) = \{v\} \cup \{v_i : 0 \leq i \leq n-1\} \cup \{u_i : 0 \leq i \leq n-1\}$, $E(H_n) = \{e_i = v_i v_{i+1} : 0 \leq i \leq n-1, \text{subscript modulo } n\} \cup \{e_i' = vv_i : 0 \leq i \leq n-1\} \cup \{e_i'' = v_i u_i : 0 \leq i \leq n-1\}$.

Theorem 5.1 *For any helm graph H_n, $\tau(L(H_n)) = n$.*

Proof By the definition of line graph, $V(L(H_n)) = \{e_i : 0 \leq i \leq n-1\} \cup \{e_i' : 0 \leq i \leq n-1\} \cup \{e_i'' : 0 \leq i \leq n-1\}$. Since e_i, e_i' and e_i'' $(0 \leq i \leq n-1)$ are adjacent edges in H_n, $\{e_i, e_i', e_i''\}$ induces a 3-cycle in $L(H_n)$ for each i, $(0 \leq i \leq n-1)$. Let $\mathbb{C} = \{e_i e_i' e_i'' : 0 \leq i \leq n-1\}$ be the set of these cycles. Clearly $\mathbb{C}$ contains vertex disjoint cycles of $L(H_n)$ and $|\mathbb{C}| = n$. Hence $\tau(L(H_n)) \geq n$. Also since $|V(L(H_n))| = 3n, \tau(L(H_n)) \leq n$. Therefore $\tau(L(H_n)) = n$. $\square$



Helm Graph H_n

Figure 5.1

Theorem 5.2 *The Tulgeity of the middle graph of the helm graph H_n, is given by*

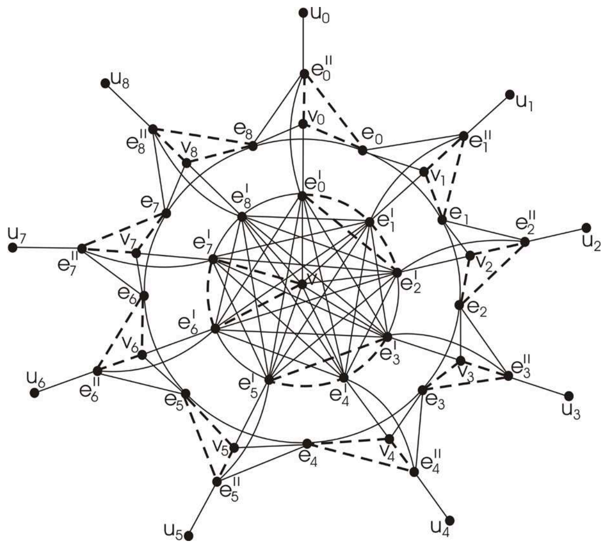
$$\tau(M(H_n)) = \left[\frac{4n+1}{3}\right].$$

Proof By the definition of middle graph, $V(M(H_n)) = V(H_n) \cup E(H_n)$, in which for each i, $(0 \le i \le n-1)$, the set of vertices $\{e_i, e_{i+1}, e'_{i+1}, e''_{i+1}, v_{i+1} : \text{subscript modulo } n\}$ induce a clique of order 5. Also $\{e'_i : 0 \le i \le n-1\} \cup \{v\}$ induces a clique of order $n+1$ (say K_{n+1}). Since $deg\, u_i = 1$ for each i, $(0 \le i \le n-1)$ in $M(H_n)$ $\tau(M(H_n)) = \tau(M(H_n) - \{u_i : 0 \le i \le n-1\})$. Hence $\tau(M(H_n)) \le \left[\frac{1}{3}(|E(H_n)| + |V(H_n)| - n)\right] = \left[\frac{4n+1}{3}\right]$. Consider the collection $\mathbb{C}$ of cycles of $M(H_n)$, $\mathbb{C} = \{v_i e_i e''_i : 0 \le i \le n-1\}$. Each cycle of $\mathbb{C}$ are vertex disjoint and $|\mathbb{C}| = n$. Also the cycles of $\mathbb{C}$ are vertex disjoint from the cycles of the clique K_{n+1}. Hence $\tau(M(H_n)) \ge |\mathbb{C}| + \left[\frac{n+1}{3}\right] = \left[\frac{4n+1}{3}\right]$. Therefore $\tau(M(H_n)) = \left[\frac{4n+1}{3}\right]$. □

Theorem 5.3 *Tulgeity of total graph of helm graph H_n, is given by*

$$\tau(T(H_n)) = \left[\frac{5n+1}{3}\right].$$

Proof By the definition of total graph, $V(T(H_n)) = V(H_n) \cup E(H_n)$ and $E(T(H_n)) = E(M(H_n)) \cup \{u_i v_i : 0 \le i \le n-1\} \cup \{v v_i : 0 \le i \le n-1\} \cup \{v_i v_{i+1} : 0 \le i \le n-1 \text{ subscripts modulo } n\}$. For each i, $(0 \le i \le n-1)$ the set of vertices $\{e_i, v_{i+1}, e_{i+1}, e'_{i+1}, e''_{i+1}\}$ of $T(H_n)$ induces a clique of order 5. Also the set of vertices $\{e'_i : 0 \le i \le n-1\} \cup \{v\}$ induces a clique K_{n+1} of order $n+1$. For each i, $(0 \le i \le n-1)$ the set of vertices $\{u_i, v_i, e''_i\}$ induces a 3-cycle in $T(H_n)$. Hence $\mathbb{C}_1 = \{u_i v_i e''_i : 0 \le i \le n-1\}$ is a set of vertex disjoint cycles of the subgraph of $T(H_n)$ induced by $\{u_i, v_i, e''_i : 0 \le i \le n-1\}$.



$M(H_9)$ and its vertex disjoint cycles

Figure 5.2

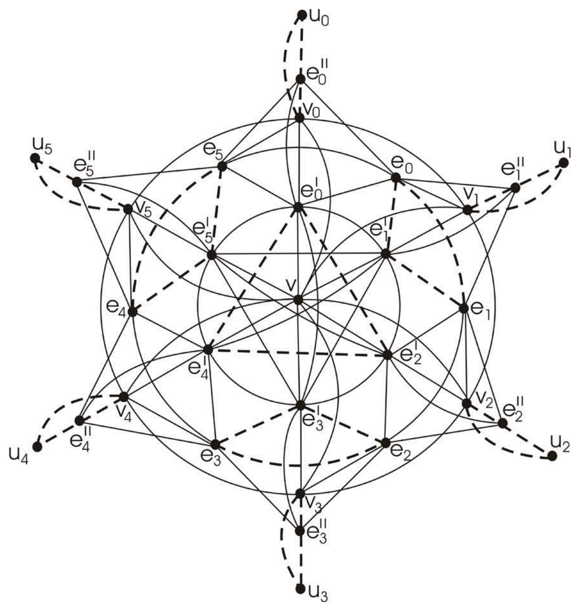
Case 1 *n is even.*

Let $\mathbb{C}_2$ be the collection of vertex disjoint 3-cycles of the subgraph induced by the set of vertices $\{e_i : 0 \le i \le n-1\} \cup \{e'_j : j = 2k+1, 0 \le k \le \dfrac{n}{2} - 1\}$. i.e., $\mathbb{C}_2 = \{e_i e_{i+1} e'_{i+1} : i = 2k, 0 \le k \le \frac{n}{2} - 1\}$. Let $\mathbb{C}_3$ be the set of 3-cycles of $T(H_n)$ induced by $\{e'_i : i = 2k, 0 \le k \le \frac{n}{2} - 1\} \cup \{v\}$. Since the subgraph induced by $\{e'_i : i = 2k, 0 \le k \le \frac{n}{2} - 1\} \cup \{v\}$ is a clique of order $\dfrac{n}{2} + 1$, $\mathbb{C}_3$ contains $\left[\dfrac{1}{3}\left(\dfrac{n}{2}+1\right)\right]$ vertex disjoint 3-cycles. Since $V(\mathbb{C}_i) \cap V(\mathbb{C}_i) = \emptyset$ for $i \ne j$, $\tau(T(H_n)) \ge |\mathbb{C}_1| + |\mathbb{C}_2| + |\mathbb{C}_3| = \left[\dfrac{5n+1}{3}\right]$.

Case 2 *n is odd.*

Let $\mathbb{C}_2 = \{e_i e_{i+1} e'_{i+1} : i = 2k, 0 \le k \le \frac{n-3}{2}\}$ be the collection of vertex disjoint cycles of the subgraph induced by $\{e_i : 0 \le i \le n-2\} \cup \{e'_i : i = 2k+1, 0 \le k \le \frac{n-3}{2}\}$. Now $V' = V(T(H_n)) - \{V(\mathbb{C}_1) \cup V(\mathbb{C}_2)\} = \{e'_{2i} : 0 \le i \le \frac{n-1}{2}\} \cup \{e_{n-1}, v\}$ has $\dfrac{5n+1}{3}$ vertices and induced subgraph $\langle V' \rangle$ contains a clique of order $\dfrac{n+3}{2}$. If $\dfrac{n+3}{2} \equiv 0$ or $1 (mod\ 3)$ then $\langle V' \rangle$ has $\left[\dfrac{1}{3}\left(\dfrac{n+5}{2}\right)\right]$ vertex disjoint 3-cycles disjoint from the cycles of $\mathbb{C}_1$ and $\mathbb{C}_2$.

If $\dfrac{n+3}{2} \equiv 2 (mod\ 3)$ then $\left\langle \{e'_{2i} : 1 \le i \le \frac{n-3}{2}\} \cup \{v\} \right\rangle$ has $\dfrac{1}{3}\left(\dfrac{n-1}{2}\right)$ vertex disjoint 3-cycles and there exists another cycle $e_{n-1} e'_{n-1} e'_0$ disjoint from the cycles of $\mathbb{C}_1, \mathbb{C}_2$ and the cycles of $\left\langle \{e'_{2i} : 1 \le i \le \frac{n-1}{2}\} \cup \{v\} \right\rangle$. Hence in both the cases $\tau(T(H_n)) \ge |\mathbb{C}_1| + |\mathbb{C}_2| + \left[\dfrac{1}{3}\left(\dfrac{n+5}{2}\right)\right] = \left[\dfrac{5n+1}{3}\right]$. Since $|V(T(H_n))| = 5n+1$, it is clear that $\tau(T(H_n)) \le \left[\dfrac{5n+1}{3}\right]$. Hence $\tau(T(H_n)) = \left[\dfrac{5n+1}{3}\right]$. $\square$



$T(H_6)$ and its vertex disjoint cycles

Figure 5.3

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Labeling, Covering and Decomposing of Graphs
— Smarandache's Notion in Graph Theory

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Abstract: This paper surveys the applications of Smarandache's notion to graph theory appeared in *International J.Math.Combin.* from Vol.1,2008 to Vol.3,2009. In fact, many problems discussed in these papers are generalized in this paper. Topics covered in this paper include: (1)What is a Smarandache System? (2)Vertex-Edge Labeled Graphs with Applications: (i)Smarandachely k-constrained labeling of a graph; (ii)Smarandachely super m-mean graph; (iii)Smarandachely uniform k-graph; (iv)Smarandachely total coloring of a graph; (3)Covering and Decomposing of a Graph: (i)Smarandache path k-cover of a graph; (ii)Smarandache graphoidal tree d-cover of a graph; (4)Furthermore.

Key Words: Smarandache system, labeled graph, Smarandachely k-constrained labeling, Smarandachely k-constrained labelingSmarandachely super m-mean graph, Smarandachely uniform k-graph, Smarandachely total coloring of a graph, Smarandache path k-cover of a graph, Smarandache graphoidal tree d-cover of a graph.

AMS(2000): 05C12, 05C70, 05C78

§1. What is a Smarandache System?

A *Smarandache System* first appeared in [1] is defined in the following.

Definition 1.1([1]) *A rule in a mathematical system $(\Sigma;\mathcal{R})$ is said to be Smarandachely denied if it behaves in at least two different ways within the same set Σ, i.e., validated and invalided, or only invalided but in multiple distinct ways.*

A Smarandache system $(\Sigma;\mathcal{R})$ is a mathematical system which has at least one Smarandachely denied rule in $\mathcal{R}$.

Definition 1.2([2]) *For an integer $m \geq 2$, let $(\Sigma_1;\mathcal{R}_1)$, $(\Sigma_2;\mathcal{R}_2)$, $\cdots$, $(\Sigma_m;\mathcal{R}_m)$ be m mathematical systems different two by two. A Smarandache multi-space is a pair $(\widetilde{\Sigma};\widetilde{\mathcal{R}})$ with*

$$\widetilde{\Sigma} = \bigcup_{i=1}^{m} \Sigma_i, \quad and \quad \widetilde{\mathcal{R}} = \bigcup_{i=1}^{m} \mathcal{R}_i.$$

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[2]Reported at Beijing Jiaotong University, November 18, 2009.

Definition 1.3([3]) *An axiom is said to be Smarandachely denied if the axiom behaves in at least two different ways within the same space, i.e., validated and invalided, or only invalided but in multiple distinct ways.*

*A Smarandache geometry is a geometry which has at least one Smarandachely denied axiom(*1969*).*

Example 1.1 Let us consider an Euclidean plane $\mathbf{R}^2$ and three non-collinear points A, B and C. Define s-points as all usual Euclidean points on $\mathbf{R}^2$ and s-lines any Euclidean line that passes through one and only one of points A, B and C, such as those shown in Fig.1.1.

(i) The axiom (A5) that through a point exterior to a given line there is only one parallel passing through it is now replaced by two statements: *one parallel*, and *no parallel*. Let L be an s-line passes through C and is parallel in the Euclidean sense to AB. Notice that through any s-point not lying on AB there is one s-line parallel to L and through any other s-point lying on AB there is no s-lines parallel to L such as those shown in Fig.1(a).

(ii) The axiom that through any two distinct points there exist one line passing through them is now replaced by; *one s-line*, and *no s-line*. Notice that through any two distinct s-points D, E collinear with one of A, B and C, there is one s-line passing through them and through any two distinct s-points F, G lying on AB or non-collinear with one of A, B and C, there is no s-line passing through them such as those shown in Fig.1(b).

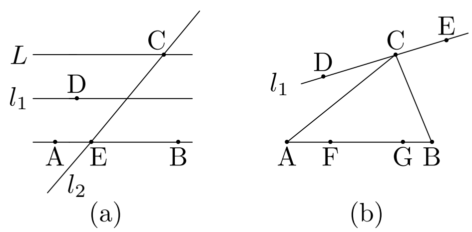

Fig.1

Definition 1.4 *A combinatorial system* $\mathscr{C}_G$ *is a union of mathematical systems* $(\Sigma_1; \mathcal{R}_1)$,$(\Sigma_2; \mathcal{R}_2)$, $\cdots$, $(\Sigma_m; \mathcal{R}_m)$ *for an integer* m, *i.e.,*

$$\mathscr{C}_G = (\bigcup_{i=1}^{m} \Sigma_i; \ \bigcup_{i=1}^{m} \mathcal{R}_i)$$

with an underlying connected graph structure G, *where*

$$V(G) = \{\Sigma_1, \Sigma_2, \cdots, \Sigma_m\},$$

$$E(G) = \{ \ (\Sigma_i, \Sigma_j) \mid \Sigma_i \bigcap \Sigma_j \neq \emptyset, 1 \leq i, j \leq m\}.$$

§2. Vertex-Edge Labeled Graphs with Applications

2.1 Application to Principal Fiber Bundles

Definition 2.1 *A labeling on a graph* $G = (V, E)$ *is a mapping* $\theta_L : V \cup E \to L$ *for a label set* L, *denoted by* G^L.

If $\theta_L : E \to \emptyset$ or $\theta_L : V \to \emptyset$, then G^L is called a vertex labeled graph or an edge labeled graph, denoted by G^V or G^E, respectively. Otherwise, it is called a vertex-edge labeled graph.

Example:

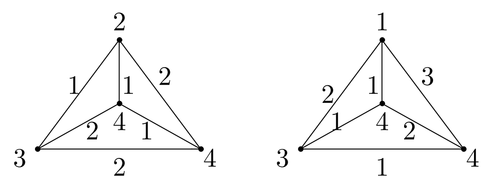

Fig.2

Definition 2.2([4]) *For a given integer sequence $0 < n_1 < n_2 < \cdots < n_m$, $m \geq 1$, a combinatorial manifold $\widetilde{M}$ is a Hausdorff space such that for any point $p \in \widetilde{M}$, there is a local chart (U_p, φ_p) of p, i.e., an open neighborhood U_p of p in $\widetilde{M}$ and a homoeomorphism $\varphi_p : U_p \to \widetilde{\mathbf{R}}(n_1(p), n_2(p), \cdots, n_{s(p)}(p))$, a combinatorial fan-space with $\{n_1(p), n_2(p), \cdots, n_{s(p)}(p)\} \subseteq \{n_1, n_2, \cdots, n_m\}$, and $\bigcup\limits_{p\in\widetilde{M}} \{n_1(p), n_2(p), \cdots, n_{s(p)}(p)\} = \{n_1, n_2, \cdots, n_m\}$, denoted by $\widetilde{M}(n_1, n_2, \cdots, n_m)$ or $\widetilde{M}$ on the context and*

$$\widetilde{\mathcal{A}} = \{(U_p, \varphi_p) | p \in \widetilde{M}(n_1, n_2, \cdots, n_m))\}$$

an atlas on $\widetilde{M}(n_1, n_2, \cdots, n_m)$.

A combinatorial manifold $\widetilde{M}$ is *finite* if it is just combined by finite manifolds with an underlying combinatorial structure G without one manifold contained in the union of others. Certainly, a finitely combinatorial manifold is indeed a combinatorial manifold. Examples of combinatorial manifolds can be seen in Fig.3.

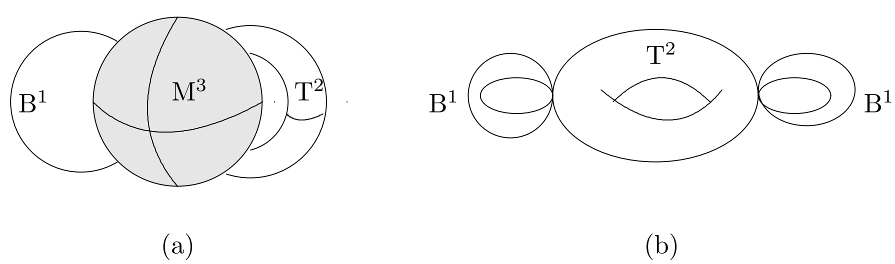

Fig.3

Let $\widetilde{M}(n_1, n_2, \cdots, n_m)$ be a finitely combinatorial manifold and $d, d \geq 1$ an integer. We construct a vertex-edge labeled graph $G^d[\widetilde{M}(n_1, n_2, \cdots, n_m)]$ by

$$V(G^d[\widetilde{M}(n_1, n_2, \cdots, n_m)]) = V_1 \bigcup V_2,$$

where $V_1 = \{n_i - \text{manifolds } M^{n_i} \text{ in } \widetilde{M}(n_1, \cdots, n_m) | 1 \leq i \leq m\}$ and $V_2 = \{\text{isolated intersection points } O_{M^{n_i}, M^{n_j}} \text{ of} M^{n_i}, M^{n_j} \text{ in } \widetilde{M}(n_1, n_2, \cdots, n_m) \text{ for } 1 \leq i, j \leq m\}$. Label n_i for each

n_i-manifold in V_1 and 0 for each vertex in V_2 and

$$E(G^d[\widetilde{M}(n_1, n_2, \cdots, n_m)]) = E_1 \bigcup E_2,$$

where $E_1 = \{(M^{n_i}, M^{n_j})$ labeled with $\dim(M^{n_i} \bigcap M^{n_j}) \mid \dim(M^{n_i} \bigcap M^{n_j}) \geq d, 1 \leq i, j \leq m\}$ and $E_2 = \{(O_{M^{n_i}, M^{n_j}}, M^{n_i}), (O_{M^{n_i}, M^{n_j}}, M^{n_j})$ labeled with $0 | M^{n_i}$ tangent M^{n_j} at the point $O_{M^{n_i}, M^{n_j}}$ for $1 \leq i, j \leq m\}$.

Now denote by $\mathcal{H}(n_1, n_2, \cdots, n_m)$ all finitely combinatorial manifolds $\widetilde{M}(n_1, n_2, \cdots, n_m)$ and $\mathcal{G}[0, n_m]$ all vertex-edge labeled graphs G^L with $\theta_L : V(G^L) \cup E(G^L) \to \{0, 1, \cdots, n_m\}$ with conditions following hold.

(1)Each induced subgraph by vertices labeled with 1 in G is a union of complete graphs and vertices labeled with 0 can only be adjacent to vertices labeled with 1.

(2)For each edge $e = (u, v) \in E(G)$, $\tau_2(e) \leq \min\{\tau_1(u), \tau_1(v)\}$.

Then we know a relation between sets $\mathcal{H}(n_1, n_2, \cdots, n_m)$ and $\mathcal{G}([0, n_m], [0, n_m])$ following.

Theorem 2.1([1]) *Let $1 \leq n_1 < n_2 < \cdots < n_m, m \geq 1$ be a given integer sequence. Then every finitely combinatorial manifold $\widetilde{M} \in \mathcal{H}(n_1, n_2, \cdots, n_m)$ defines a vertex-edge labeled graph $G([0, n_m]) \in \mathcal{G}[0, n_m]$. Conversely, every vertex-edge labeled graph $G([0, n_m]) \in \mathcal{G}[0, n_m]$ defines a finitely combinatorial manifold $\widetilde{M} \in \mathcal{H}(n_1, n_2, \cdots, n_m)$ with a $1-1$ mapping $\theta : G([0, n_m]) \to \widetilde{M}$ such that $\theta(u)$ is a $\theta(u)$-manifold in $\widetilde{M}$, $\tau_1(u) = \dim\theta(u)$ and $\tau_2(v, w) = \dim(\theta(v) \bigcap \theta(w))$ for $\forall u \in V(G([0, n_m]))$ and $\forall (v, w) \in E(G([0, n_m]))$.*

Definition 2.3([4]) *A principal fiber bundle consists of a manifold P action by a Lie group $\mathscr{G}$, which is a manifold with group operation $\mathscr{G} \times \mathscr{G} \to \mathscr{G}$ given by $(g, h) \to g \circ h$ being C^∞ mapping, a projection $\pi : P \to M$, a base pseudo-manifold M, denoted by $(P, M, \mathscr{G})$, seeing Fig.4 (where $V = \pi^{-1}(U)$) such that conditions (1), (2) and (3) following hold.*

(1) there is a right freely action of $\mathscr{G}$ on P,, i.e., for $\forall g \in \mathscr{G}$, there is a diffeomorphism $R_g : P \to P$ with $R_g(p) = pg$ for $\forall p \in P$ such that $p(g_1 g_2) = (pg_1)g_2$ for $\forall p \in P$, $\forall g_1, g_2 \in \mathscr{G}$ and $pe = p$ for some $p \in P$, $e \in \mathscr{G}$ if and only if e is the identity element of $\mathscr{G}$.

(2) the map $\pi : P \to M$ is onto with $\pi^{-1}(\pi(p)) = \{pg | g \in \mathscr{G}\}$.

(3) for $\forall x \in M$ there is an open set U with $x \in U$ and a diffeomorphism $T_U : \pi^{-1}(U) \to U \times \mathscr{G}$ of the form $T_U(p) = (\pi(p), s_U(p))$, where $s_U : \pi^{-1}(U) \to \mathscr{G}$ has the property $s_U(pg) = s_U(p)g$ for $\forall g \in \mathscr{G}, p \in \pi^{-1}(U)$.

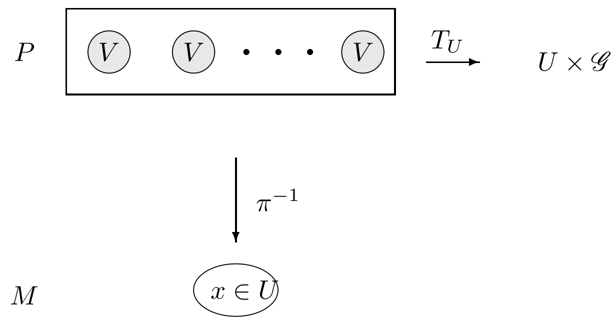

Fig.4

Question *For a family of k principal fiber bundles $P_1(M_1,\mathscr{G}_1)$, $P_2(M_2,\mathscr{G}_2)$,$\cdots$, $P_k(M_k,\mathscr{G}_k)$ over manifolds M_1, M_2, $\cdots$, M_k, how can we construct principal fiber bundles on a smoothly combinatorial manifold consisting of $M_1, M_2, \cdots, M_k$ underlying a connected graph G?*

The answer is YES. The technique is by voltage assignment on labeled graphs defined as follows.

Definition 2.4([4]) *A voltage labeled graph on a vertex-edge labeled graph G^L is a 2-tuple $(G^L;\alpha)$ with a voltage assignments $\alpha : E(G^L) \to \Gamma$ such that*

$$\alpha(u,v) = \alpha^{-1}(v,u), \quad \forall (u,v) \in E(G^L),$$

with its labeled lifting G^{L_α} defined by

$$V(G^{L_\alpha}) = V(G^L) \times \Gamma, \ \ (u,g) \in V(G^L) \times \Gamma \ \textit{abbreviated to } u_g;$$

$$E(G^L_\alpha) = \{\ (u_g, v_{g\circ h}) \mid \text{for } \ \forall (u,v) \in E(G^L) \text{ with } \alpha(u,v) = h\ \}$$

with labels $\Theta_L : G^{L_\alpha} \to L$ following:

$$\Theta_L(u_g) = \theta_L(u), \ \text{ and } \ \Theta_L(u_g, v_{g\circ h}) = \theta_L(u,v)$$

for $u, v \in V(G^L)$, $(u,v) \in E(G^L)$ with $\alpha(u,v) = h$ and $g, h \in \Gamma$.

For a voltage labeled graph (G^L, α) with its lifting G^L_α, a *natural projection* $\pi : G^{L_\alpha} \to G^L$ is defined by $\pi(u_g) = u$ and $\pi(u_g, v_{g\circ h}) = (u,v)$ for $\forall u, v \in V(G^L)$ and $(u,v) \in E(G^L)$ with $\alpha(u,v) = h$. Whence, (G^{L_α}, π) is a covering space of the labeled graph G^L. A voltage labeled graph with its labeled lifting are shown in Fig.4.4, in where, $G^L = C_3^L$ and $\Gamma = Z_2$.

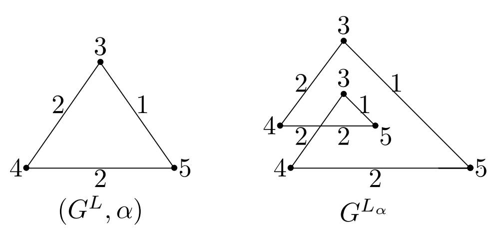

Fig.5

Now we show how to construct principal fiber bundles over a combinatorial manifold $\widetilde{M}$.

Construction 2.1 *For a family of principal fiber bundles over manifolds* $M_1, M_2, \cdots, M_l$, *such as those shown in Fig.*6,

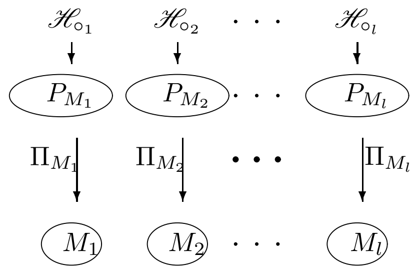

Fig.6

where $\mathscr{H}_{\circ_i}$ *is a Lie group acting on* P_{M_i} *for* $1 \leq i \leq l$ *satisfying conditions PFB1-PFB3, let* $\widetilde{M}$ *be a differentiably combinatorial manifold consisting of* M_i, $1 \leq i \leq l$ *and* $(G^L[\widetilde{M}], \alpha)$ *a voltage graph with a voltage assignment* $\alpha : G^L[\widetilde{M}] \to \mathfrak{G}$ *over a finite group* $\mathfrak{G}$, *which naturally induced a projection* $\pi : G^L[\widetilde{P}] \to G^L[\widetilde{M}]$. *For* $\forall M \in V(G^L[\widetilde{M}])$, *if* $\pi(P_M) = M$, *place* P_M *on each lifting vertex* M^{L_α} *in the fiber* $\pi^{-1}(M)$ *of* $G^{L_\alpha}[\widetilde{M}]$, *such as those shown in Fig.*7.

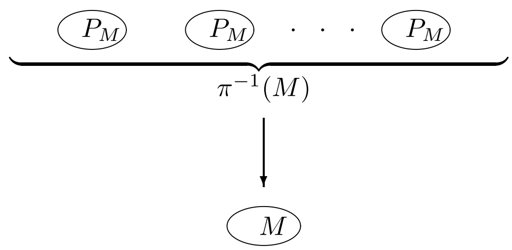

Fig.7

Let $\Pi = \pi \Pi_M \pi^{-1}$ *for* $\forall M \in V(G^L[\widetilde{M}])$. *Then* $\widetilde{P} = \bigcup\limits_{M \in V(G^L[\widetilde{M}])} P_M$ *is a smoothly combinatorial manifold and* $\mathscr{L}_G = \bigcup\limits_{M \in V(G^L[\widetilde{M}])} \mathscr{H}_M$ *a Lie multi-group by definition. Such a constructed combinatorial fiber bundle is denoted by* $\widetilde{P}^{L_\alpha}(\widetilde{M}, \mathscr{L}_G)$.

For example, let $\mathfrak{G} = Z_2$ and $G^L[\widetilde{M}] = C_3$. A voltage assignment $\alpha : G^L[\widetilde{M}] \to Z_2$ and its induced combinatorial fiber bundle are shown in Fig.8.

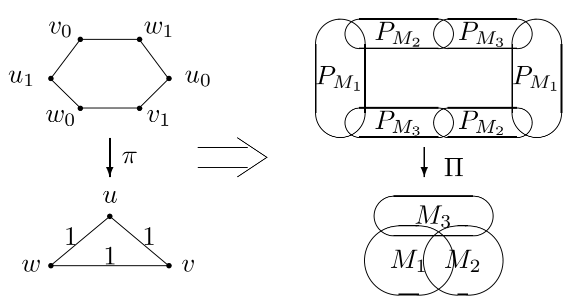

Fig.8

Then we know the existence result following.

Theorem 2.2([4]) *A combinatorial fiber bundle $\widetilde{P}^{\alpha}(\widetilde{M},\mathscr{L}_G)$ is a principal fiber bundle if and only if for* $\forall(M',M'')\in E(G^L[\widetilde{M}])$ *and* $(P_{M'},P_{M''})=(M',M'')^{L_\alpha}\in E(G^L[\widetilde{P}])$, $\Pi_{M'}|_{P_{M'}\cap P_{M''}}=\Pi_{M''}|_{P_{M'}\cap P_{M''}}$.

2.2 Smarandachely k-constrained labeling of a graph

In references [5]-[6], the Smarandachely k-constrained labeling on some graph families are discussed.

Definition 2.5 *A Smarandachely k-constrained labeling of a graph $G(V,E)$ is a bijective mapping $f: V\cup E\rightarrow\{1,2,..,|V|+|E|\}$ with the additional conditions that $|f(u)-f(v)|\geq k$ whenever $uv\in E$, $|f(u)-f(uv)|\geq k$ and $|f(uv)-f(vw)|\geq k$ whenever $u\neq w$, for an integer $k\geq 2$. A graph G which admits a such labeling is called a Smarandachely k-constrained total graph, abbreviated as $k-CTG$.*

An example for $k=5$:

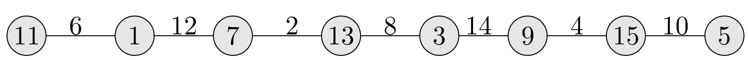

Fig.9: A 5-constrained labeling of a path P_7.

Definition 2.6 *The minimum positive integer n such that the graph $G\cup\overline{K}_n$ is a $k-CTG$ is called k-constrained number of the graph G and denoted by $t_k(G)$, the corresponding labeling is called a minimum k-constrained total labeling of G.*

Problem 2.1 *Determine $t_k(G)$ for $\forall k\in\mathbf{Z}^+$ and a graph G.*

$\gg$Update Results for Problem 2.1 obtained in [5]-[6]:

Case 1. $k=1$

In fact, $t_1(G)=0$ for any graph G since any bijective mapping $f: V\cup E\rightarrow\{1,2,..,|V|+|E|\}$ satisfies that $|f(u)-f(v)|\geq 1$ whenever $uv\in E$, $|f(u)-f(uv)|\geq 1$ and $|f(uv)-f(vw)|\geq 1$ whenever $u\neq w$.

Case 2. $k=2$

(1) $t_2(P_n)=\begin{cases}0 & if\quad n=2,\\ 1 & if\quad n=3,\\ 0 & else.\end{cases}$

Proof Let $V(P_n)=\{v_1,v_2,...,v_n\}$ and $E(P_n)=\{v_iv_{i+1}|1\leq i\leq n-1\}$. Consider a total labeling $f: V\cup E\longrightarrow\{1,2,3,...,2n-1\}$ defined as $f(v_1)=2n-3$; $f(v_2)=2n-1$; $f(v_1v_2)=2$; $f(v_2v_3)=4$; and $f(v_k)=2k-5$, $f(v_kv_{k+1})=2k$, for all $k\geq 3$. This function f serves as a Smarandachely 2-constrained labeling for P_n, for $n\geq 4$. Further, the cases $n=2$ and $n=3$

are easy to prove. □

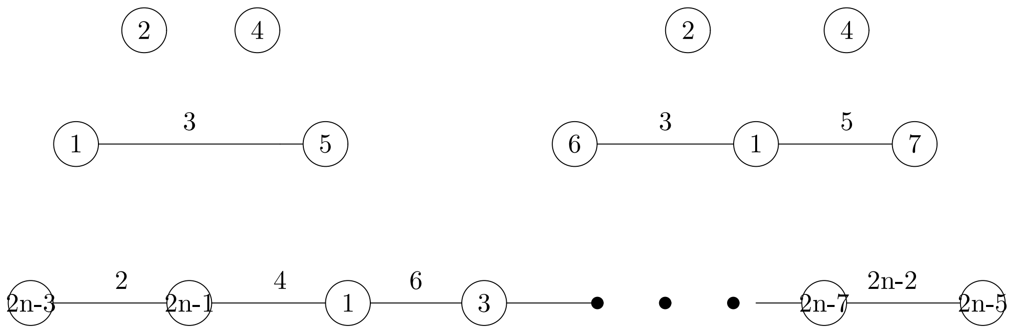

Fig.10

(2) $t_2(C_n) = 0$ if $n \geq 4$ and $t_2(C_3) = 2$.

Proof If $n \geq 4$, then the result follows immediately by joining end vertices of P_n by an edge v_1v_n , and, extending the total labeling f of the path as in the proof of the Theorem 2.4 above to include $f(v_1v_2) = 2n$.

Consider the case $n = 3$. If the integers a and $a+1$ are used as labels, then one of them is assigned for a vertex and other is to the edge not incident with that vertex. But then, $a+2$ can not be used to label the vertex or an edge in C_3. Therefore, for each three consecutive integers we should leave at least one integer to label C_3. Hence the span of any Smarandachely 2-constrained labeling of C_3 should be at least 8. So $t_2(C_3) \geq 2$. Now from the Figure 3 it is clear that $t_2(C_3) \leq 2$. Thus $t_2(C_3) = 2$. □

(3) $t_2(K_n) = 0$ if $n \geq 4$.

(4) $t_2(W_{1,n}) = 0$ if $n \geq 3$.

(6) $t_2(K_{m,n})=\begin{cases} 2 \;\; if \;\; n=1 \;\; and \;\; m=1, \\ 1 \;\; if \;\; n=1 \;\; and \;\; m \geq 2, \\ 0 \;\; else. \end{cases}$

Case 3. $k \geq 3$

(1) $t_k(K_{1,n})=\begin{cases} 3k-6, \;\; if \;\; n=3, \\ n(k-2), \;\; otherwise. \end{cases}$ if $k.n \geq 3$.

Proof For any Smarandachely k-constrained labeling f of a star $K_{1,n}$, the span of f, after labeling an edge by the least positive integer a is at least $a+nk$. Further, the span is minimum only if $a = 1$. Thus, as there are only $n+1$ vertices and n edges, for any minimum total labeling we require at least $1+nk-(2n+1) = n(k-2)$ isolated vertices if $n \geq 4$ and at least $1+nk-2n = n(k-2)+1$ if $n = 3$. In fact, for the case $n = 3$, as the central vertex is incident with each edge and edges are mutually adjacent, by a minimum k-constrained total labeling, the edges as well the central vertex can be labeled only by the set $\{1, 1+k, 1+2k, 1+3k\}$.

Suppose the label 1 is assigned for the central vertex, then to label the end vertex adjacent to edge labeled $1+2k$ is at least $(1+3k)+1$ (since it is adjacent to 1, it can not be less than $1+k$). Thus at most two vertices can only be labeled by the integers between 1 and $1+3k$. Similar argument holds for the other cases also.

Therefore, $t(K_{1,n}) \geq n(k-2)$ for $n \geq 4$ and $t(K_{1,n}) \geq n(k-2)+1$ for $n=3$.

To prove the reverse inequality, we define a k-constrained total labeling for all $k \geq 3$, as follows:

(1) When $n=3$, the labeling is shown in the Fig.11 below

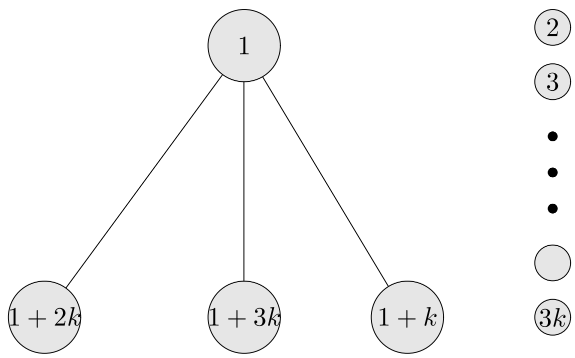

Fig.11

(2) When $n \geq 4$, define a total labeling f as $f(v_0v_j) = 1+(j-1)k$ for all $j, 1 \leq j \leq n$. $f(v_0) = 1+nk$, $f(v_1) = 2+(n-2)k$, $f(v_2) = 3+(n-2)k$,and for $3 \leq i \leq (n-1)$,

$$f(v_{i+1}) = \begin{cases} f(v_i)+2, & if \quad f(v_i) \equiv 0 (mod \quad k), \\ f(v_i)+1, & otherwise. \end{cases}$$

and the rest all unassigned integers between 1 and $1+nk$ to the $n(k-2)$ isolated vertices, where v_0 is the central vertex and $v_1, v_2, v_3, ..., v_n$ are the end vertices.

The function so defined is a Smarandachely k-constrained labeling of $K_{1,n} \cup \bar{K}_{n(k-2)}$, for all $n \geq 4$. □

(2) Let P_n be a path on n vertices and $k_0 = \lfloor \frac{2n-1}{3} \rfloor$. Then

$$t_k(P_n) = \begin{cases} 0 \quad if \quad k \leq k_0, \\ 2(k-k_0)-1 \quad if \quad k > k_0 \quad and \quad 2n \equiv 0 (mod\ 3), \\ 2(k-k_0) \quad if \ k > k_0 \quad and \quad 2n \equiv 1 \ or \ 2 (mod\ 3). \end{cases}$$

(3) Let C_n be a cycle on n vertices and $k_0 = \lfloor \frac{2n-1}{3} \rfloor$. Then

$$t_k(C_n) = \begin{cases} 0 \quad if \quad k \leq k_0, \\ 2(k-k_0) \quad if \quad k > k_0 \quad and \quad 2n \equiv 0 \ (mod\ 3), \\ 3(k-k_0) \quad if \ k > k_0 \quad and \quad 2n \equiv 1 \ or \ 2 (mod\ 3). \end{cases}$$

2.3 Smarandachely Super m-Mean Graph

The conception of Smarandachely edge m-labeling on a graph was introduced in [7].

Definition 2.7 *Let G be a graph and $f : V(G) \to \{1, 2, 3, \cdots, |V| + |E(G)|\}$ be an injection. For each edge $e = uv$ and an integer $m \geq 2$, the induced Smarandachely edge m-labeling f_S^* is defined by*

$$f_S^*(e) = \left\lceil \frac{f(u) + f(v)}{m} \right\rceil.$$

Then f is called a Smarandachely super m-mean labeling if $f(V(G)) \cup \{f^*(e) : e \in E(G)\} = \{1, 2, 3, \cdots, |V| + |E(G)|\}$. A graph that admits a Smarandachely super mean m-labeling is called Smarandachely super m-mean graph.

Particularly, if $m = 2$, we know that

$$f^*(e) = \begin{cases} \frac{f(u)+f(v)}{2} & \text{if } f(u) + f(v) \text{ is even;} \\ \frac{f(u)+f(v)+1}{2} & \text{if } f(u) + f(v) \text{ is odd.} \end{cases}$$

Example: A Smarandache super 2-mean graph P_6^2

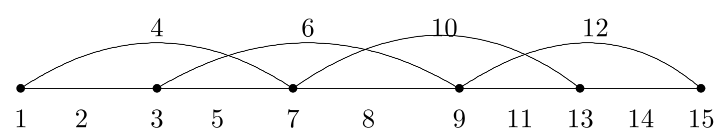

Fig.12

Problem 2.2 *Find integers m and graphs G such that G is a Smarandachely super m-mean graph.*

≫Update Results for Problem 2.2 **Obtained in [7]:**

Now all results is on the case of Smarandache super 2-mean graphs.

(1) A *H-graph* of a path P_n is the graph obtained from two copies of P_n with vertices $v_1, v_2, \ldots, v_n$ and $u_1, u_2, \ldots, u_n$ by joining the vertices $v_{\frac{n+1}{2}}$ and $u_{\frac{n+1}{2}}$ if n is odd and the vertices $v_{\frac{n}{2}+1}$ and $u_{\frac{n}{2}}$ if n is even. Then

A H-graph G is a Smarandache super 2-mean graph.

(2) The corona of a graph G on p vertices $v_1, v_2, \ldots, v_p$ is the graph obtained from G by adding p new vertices $u_1, u_2, \ldots, u_p$ and the new edges u_iv_i for $1 \leq i \leq p$, denoted by $G \odot K_1$.

If a H-graph G is a Smarandache super 2-mean graph, then $G \odot K_1$ is a Smarandache super 2-mean graph.

(3) For a graph G, the 2-corona of G is the graph obtained from G by identifying the center vertex of the star S_2 at each vertex of G, denoted by $G \odot S_2$.

If a H-graph G is a Smarandache super 2-mean graph, then $G \odot S_2$ is a Smarandache super 2-mean graph.

(4) *Cycle C_{2n} is a Smarandache super 2-mean graph for $n \geq 3$.*

(5) *Corona of a cycle C_n is a Smarandache super 2-mean graph for $n \geq 3$.*

(6) A cyclic snake mC_n is the graph obtained from m copies of C_n by identifying the vertex $v_{(k+2)_j}$ in the j^{th} copy at a vertex $v_{1_{j+1}}$ in the $(j+1)^{th}$ copy if $n = 2k+1$ and identifying the vertex $v_{(k+1)_j}$ in the j^{th} copy at a vertex $v_{1_{j+1}}$ in the $(j+1)^{th}$ copy if $n = 2k$.

The graph mC_n-snake, $m \geq 1, n \geq 3$ and $n \neq 4$ has a Smarandache super 2-mean labeling.

(7) A $P_n(G)$ is a graph obtained from G by identifying an end vertex of P_n at a vertex of G.

If G is a Smarandache super 2-mean graph then $P_n(G)$ is also a Smarandache super 2-mean graph.

(8) *$C_m \times P_n$ for $n \geq 1, m = 3, 5$ are Smarandache super 2-mean graphs.*

Problem 2.3 *For what values of m (except 3,5) the graph $C_m \times P_n$ is a Smarandache super 2-mean graph?*

2.4 **Smarandachely Uniform k-Graphs**

The conception of Smarandachely Uniform k-Graph was introduced in the reference [8].

Definition 2.7 *For an non-empty subset M of vertices in a graph $G = (V, E)$, each vertex u in G is associated with the set $f_M^o(u) = \{d(u, v) \ : \ v \in M, \ u \neq v\}$, called its open M-distance-pattern.*

A graph G is called a Smarandachely uniform k-graph if there exist subsets $M_1, M_2, \cdots, M_k$ for an integer $k \geq 1$ such that $f_{M_i}^o(u) = f_{M_j}^o(u)$ and $f_{M_i}^o(u) = f_{M_j}^o(v)$ for $1 \leq i, j \leq k$ and $\forall u, v \in V(G)$. Such subsets $M_1, M_2, \cdots, M_k$ are called a k-family of open distance-pattern uniform (odpu-) set of G and the minimum cardinality of odpu-sets in G, if they exist, is called the Smarandachely odpu-number of G, denoted by $od_k^S(G)$.

Usually, a Smarandachely uniform 1-graph G is called an *open distance-pattern uniform (odpu-) graph*. In this case, its odpu-number $od_k^S(G)$ of G is abbreviated to $od(G)$.

Problem 2.4 *Determine which graph G is Smarandachely uniform k-graph for an integer $k \geq 1$.*

$\gg$Update Results for Problem 2.4 Obtained in [8]:

(1) *A connected graph G is an odpu-graph if and only if the center $Z(G)$ of G is an odpu-set.*

(2) *Every self-centered graph is an odpu-graph.*

(3) *A tree T has an odpu-set M if and only if T is isomorphic to P_2.*

(4) *If G is a unicyclic odpu-graph, then G is isomorphic to a cycle.*

(5) *A block graph G is an odpu-graph if and only if G is complete.*

(6) *A graph with radius 1 and diameter 2 is an odpu-graph if and only if there exists a subset $M \subset V(G)$ with $|M| \geq 2$ such that the induced subgraph $\langle M \rangle$ is complete, $\langle V - M \rangle$ is not complete and any vertex in $V - M$ is adjacent to all the vertices of M.*

Problem 2.5 *Determine the Smarandachely odpu-number $od_k^S(G)$ of G for an integer $k \geq 1$.*

$\gg$**Update Results for Problem** 2.5 **obtained in [8]:**

(1) *For every positive integer $k \neq 1, 3$, there exists a graph G with odpu-number k.*

(2) *If a graph G has odpu-number 4, then $r(G) = 2$.*

(3) *The number 5 cannot be the odpu-number of a bipartite graph.*

(4) *Let G be a bipartite odpu-graph. Then $od(G) = 2$ if and only if G is isomorphic to P_2.*

(5) $od(C_{2k+1}) = 2k$.

(6) $od(K_n) = 2$ *for all* $n \geqslant 2$.

2.5 Smarandachely Total Coloring of a graph

The conception of Smarandachely total k-coloring of a graph following is introduced by Zhongfu Zhang et al. in [9].

Definition 2.8 *Let f be a total $k-$coloring on G. Its total-color neighbor of a vertex u of G is denoted by $C_f(x) = \{f(x)|x \in T_N(u)\}$. For any adjacent vertices x and y of $V(G)$, if $C_f(x) \neq C_f(y)$, say f a k AVSDT-coloring of G (the abbreviation of adjacent-vertex-strongly-distinguishing total coloring of G).*

The AVSDT-coloring number of G, denoted by $\chi_{ast}(G)$ is the minimal number of colors required for an AVSDT-coloring of G

Definition 2.9 *A Smarandachely total k-coloring of a graph G is an AVSDT-coloring with $|C_f(x)\backslash C_f(y)| \geq k$ and $|C_f(y)\backslash C_f(x)| \geq k$.*

The minimum Smarandachely total k-coloring number of a graph G is denoted by $\chi_{ast}^k(G)$.

Obviously, $\chi_{ast}(G) = \chi_{ast}^1(G)$ and

$$\cdots \leq \chi_{ast}^{k+1}(G) \leq \chi_{ast}^k(G) \leq \chi_{ast}^{k-1}(G) \leq \cdots \leq \chi_{ast}^1(G)$$

by definition.

Problem 2.6 Determine $\chi_{ast}^k(G)$ for a graph G.

$\gg$**Update Results for Problem** 2.6 **obtained in [9]:**

$\chi_{ast}^1(S_m + W_n) = m + n + 3$ *if* $min\{m, n\} \geq 5$.

It should be noted that the number $\chi^k_{ast}(G)$ of graph families following are determined for integers $k \geq 1$ by Zhongfu Zhang et al. in references [10]-[15].

(1) 3-regular Halin graphs;

(2) $2P_n,\ 2C_n,\ 2K_{1,n}$ and double fan graphs for integers $n \geq 1$;

(3) $P_m + P_n$ for integers $m, n \geq 1$;

(4) $P_m \vee P_n$ for integers $m, n \geq 1$;

(5) Generalized Petersen $G(n, k)$;

(6) k-cube graphs.

§3. Covering and Decomposing of a Graph

Definition 3.1 *Let $\mathscr{P}$ be a graphical property. A Smarandache graphoidal $\mathscr{P}$ (k, d)-cover of a graph G is a partition of edges of G into subgraphs $G_1, G_2, \cdots, G_l \in \mathscr{P}$ such that $E(G_i) \cap E(G_j) \leq k$ and $\Delta(G_i) \leq d$ for integers $1 \leq i, j \leq l$.*

The minimum cardinality of Smarandache graphoidal $\mathscr{P}$ (k, d)-cover of a graph G is denoted by $\Pi^{(k,d)}_{\mathscr{P}}(G)$.

Problem 3.1 *determine $\Pi^{(k,d)}_{\mathscr{P}}(G)$ for a graph G.*

3.1 Smarandache path k-cover of a graph

The Smarandache path k-cover of a graph was discussed by S. Arumugam and I.Sahul Hamid in [16].

Definition 3.2 *A Smarandache path k-cover of a graph G is a Smarandache graphoidal $\mathscr{P}$ $(k, \Delta(G))$-cover of G with $\mathscr{P}$=path for an integer $k \geq 1$.*

A Smarandache path 1-cover of G such that its every edge is in exactly one path in it is called a simple path cover.

The minimum cardinality of simple path covers of G is called the simple path covering number of G and is denoted by $\Pi^{(1,\Delta(G))}_{\mathscr{P}}(G)$.

If do not consider the condition $E(G_i) \cap E(G_j) \leq 1$, then a simple path cover is called path cover of G, its minimum number of path cover is denoted by $\pi(G)$ in reference. For examples, $\pi_s(K_n) = \lceil \frac{n}{2} \rceil$ and $\pi_s(T) = \frac{k}{2}$, where k is the number of odd degree in tree T.

Problem 3.2 *determine $\Pi^{(k,d)}_{\mathscr{P}}(G)$ for a graph G.*

$\gg$Update Results for Problem 3.2 Obtained in [10]:

(1) $\Pi^{(1,\Delta(G))}_{\mathscr{P}}(T) = \pi(T) = \frac{k}{2}$, *where k is the number of vertices of odd degree in T.*

(2) *Let G be a unicyclic graph with cycle C. Let m denote the number of vertices of degree greater than 2 on C. Let k be the number of vertices of odd degree. Then*

$$\Pi_{\mathscr{P}}^{(1,\Delta(G))}(G) = \begin{cases} 3 & if\ m = 0 \\ \frac{k}{2} + 2 & if\ m = 1 \\ \frac{k}{2} + 1 & if\ m = 2 \\ \frac{k}{2} & if\ m \geq 3 \end{cases}$$

(3) *For a wheel* $W_n = K_1 + C_{n-1}$*, we have*

$$\Pi_{\mathscr{P}}^{(1,\Delta(G))}(W_n) = \begin{cases} 6 & if\ n = 4 \\ \left\lfloor \frac{n}{2} \right\rfloor + 3 & if\ n \geq 5 \end{cases}$$

Proof Let $V(W_n) = \{v_0, v_1, \ldots, v_{n-1}\}$ and $E(W_n) = \{v_0 v_i : 1 \leq i \leq n-1\} \cup \{v_i v_{i+1} : 1 \leq i \leq n-2\} \cup \{v_1 v_{n-1}\}$.

If $n = 4$, then $W_n = K_4$ and hence $\Pi_{\mathscr{P}}^{(1,\Delta(G))}(W_n)(W_n) = 6$.

Now, suppose $n \geq 5$. Let $r = \left\lfloor \frac{n}{2} \right\rfloor$

If n is odd, let

$P_i = (v_i, v_0, v_{r+i})$, $i = 1, 2, \ldots, r$.

$P_{r+1} = (v_1, v_2, \ldots, v_r)$,

$P_{r+2} = (v_1, v_{2r}, v_{2r-1}, \ldots, v_{r+2})$ and

$P_{r+3} = (v_r, v_{r+1}, v_{r+2})$.

If n is even, let

$P_i = (v_i, v_0, v_{r-1+i})$, $i = 1, 2, \ldots, r-1$.

$P_r = (v_0, v_{2r-1})$,

$P_{r+1} = (v_1, v_2, \ldots, v_{r-1})$,

$P_{r+2} = (v_1, v_{2r-1}, \ldots, v_{r+1})$ and

$P_{r+3} = (v_{r-1}, v_r, v_{r+1})$.

Then $\Pi_{\mathscr{P}}^{(1,\Delta(G))}(W_n) = \{P_1, P_2, \ldots, P_{r+3}\}$ is a simple path cover of W_n. Hence $\pi_s(W_n) \leq r + 3 = \left\lfloor \frac{n}{2} \right\rfloor + 3$. Further, for any simple path cover ψ of W_n at least three vertices on $C = (v_1, v_2, \ldots, v_{n-1})$ are terminal vertices of paths in ψ. Hence $t \leq q - \frac{k}{2} - 3$, so that $\Pi_{\mathscr{P}}^{(1,\Delta(G))}(W_n) = q - t \geq \frac{k}{2} + 3 = \left\lfloor \frac{n}{2} \right\rfloor + 3$. Thus $\Pi_{\mathscr{P}}^{(1,\Delta(G))}(W_n) = \left\lfloor \frac{n}{2} \right\rfloor + 3$. □

A. Nagarajan, V. Maheswari and S. Navaneethakrishnan discussed Smarandache path 1-cover in [17].

Definition 3.3 *A Smarandache path* 1*-cover of* G *such that its every edge is in exactly two path in it is called a path double cover.*

Define $G * H$ with vertex set $V(G) \times V(H)$ in which (g_1, h_1) is joined to (g_2, h_2) whenever $g_1 g_2 \in E(G)$ or $g_1 = g_2$ and $h_1 h_2 \in E(H)$; $G \circ H$, the weak product of graphs G, H with vertex

set $V(G)\times V(H)$ in which two vertices (g_1,h_1) and (g_2,h_2) are adjacent whenever $g_1g_2\in E(G)$ and $h_1h_2\in E(H)$ and

$$\gamma_2(G)=\min\{\ |\psi|:\psi \text{ is a path double cover of } G\ \}.$$

(4) *Let $m\geq 3$.*

$$\gamma_2(C_m\circ K_2)=\begin{cases}3 & \text{if m is odd;}\\ 6 & \text{if m is even.}\end{cases}$$

(5) *Let $m,n\geq 3$. $\gamma_2(C_m\circ C_n)=5$ if at least one of the numbers m and n is odd.*

(6) *Let $m,n\geq 3$.*

$$\gamma_2(P_m\circ C_n)=\begin{cases}4 & \text{if n }\equiv 1 \text{ or } 3(\text{mod } 4)\\ 8 & \text{if n }\equiv 0 \text{ or } 2(\text{mod } 4)\end{cases}$$

(7) $\gamma_2(C_m*K_2)=6$ *if $m\geq 3$ is odd.*

(8) $\gamma_2(P_m*K_2)=4$ *for $m\geq 3$.*

(9) $\gamma_2(P_m*K_2)=5$ *for $m\geq 3$.*

(10) $\gamma_2(C_m\times P_3)=5$ *if $m\geq 3$ is odd.*

(11) $\gamma_2(P_m\circ K_2)=4$ *for $m\geq 2$.*

(12) $\gamma_2(K_{m,n})=\max\{m,n\}$.

(13)

$$\gamma_2(P_m\times P_n)=\begin{cases}3 & \text{if m=2 or n=2;}\\ 4 & \text{if } m,n\geq 2.\end{cases}$$

(14) $\gamma_2(C_m\times C_n)=5$ *if $m\geq 3$, $n\geq 3$ and at least one of the numbers m and n is odd.*

(15) $\gamma_2(C_m\times K_2)=4$ *for $m\geq 3$.*

3.2 Smarandache graphoidal tree d-cover of a graph

S.Somasundaram, A.Nagarajan and G.Mahadevan discussed Smarandache graphoidal tree d-cover of a graph in references [18]-[19].

Definition 3.4 *A Smarandache graphoidal tree d-cover of a graph G is a Smarandache graphoidal $\mathscr{P}$ $(|G|,d)$-cover of G with $\mathscr{P}$=tree for an integer $d\geq 1$.*

The minimum cardinality of Smarandache graphoidal tree d-cover of G is denoted by $\gamma_T^{(d)}(G)=\Pi_{\mathscr{P}}^{(|G|,d)}(G)$. If $d=\Delta(G)$, then $\gamma_T^{(d)}(G)$ is abbreviated to $\gamma_T(G)$.

Problem 3.3 *determine $\gamma_T(G)$ for a graph G, particularly, $\gamma_T(G)$.*

$\gg$Update Results for Problem 3.3 Obtained in [12-13]:

Case 1: $\gamma_T(G)$

(1) $\gamma_T(K_p)=\lceil\frac{p}{2}\rceil$;

(2) $\gamma_T(K_{m,n})=\lceil\frac{m+n}{3}\rceil$ *if $m\leq n<2m-3$.*

(3) $\gamma_T(K_{m,n})=m$ *if $n>2m-3$.*

(4) $\gamma_T(P_m \times P_n) = 2$ *for integers* $m, n \geq 2$.

(5) $\gamma_T(P_n \times C_m) = 2$ *for integers* $m \geq 3$, $n \geq 2$.

(6) $\gamma_T(C_m \times C_n) = 3$ *if* $m, n \geq 3$.

Case 2: $\gamma_T^{(d)}(G)$

(1)

$$\gamma_T^{(d)}(K_p) = \begin{cases} \frac{p(p-2d+1)}{2} & if\ d < \frac{p}{2}, \\ \lceil \frac{p}{2} \rceil & if\ d \geq \frac{p}{2} \end{cases}$$

if $p \geq 4$.

(2) $\gamma_T^{(d)}(K_{m,n}) = p + q - pd = mn - (m+n)(d-1)$ *if* $n, m \geq 2d$.

(3) $\gamma_T^{(d)}(K_{2d-1,2d-1}) = p + q - pd = 2d - 1$.

(4) $\gamma_T^{(d)}(K_{n,n}) = \lceil \frac{2n}{3} \rceil$ *for* $d \geq \lceil \frac{2n}{3} \rceil$ *and* $n > 3$.

(5) $\gamma_T^{(d)}(C_m \times C_n) = 3$ *for* $d \geq 4$ *and* $\gamma_T^{(2)}(C_m \times C_n) = q - p$.

§5. Furthermore

In fact, Smarandache's notion can be used to generalize more and more conceptions and problems in classical graph theory. Some of them will appeared in my books *Automorphism Groups of Maps, Surfaces and Smarandache's Geometries* (Second edition), *Smarandache Multi-Space Theory* (Second edition) published in forthcoming, or my monograph *Graph Theory – A Smarandachely Type* will be appeared in 2012.

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[12]W.S. Massey, *Algebraic topology: an introduction*, Springer-Verlag, New York 1977.

Research papers

[6]Linfan Mao, Combinatorial speculation and combinatorial conjecture for mathematics, *International J.Math. Combin.*, Vol.1, 1-19(2007).

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